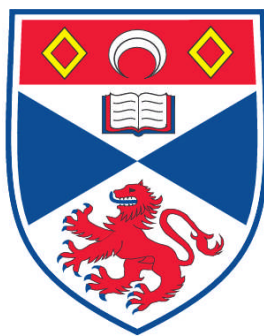


APPLICATION OF STOCHASTIC DIFFERENTIAL GAMES AND REAL OPTION THEORY IN ENVIRONMENTAL ECONOMICS

Wen-Kai Wang

**A Thesis Submitted for the Degree of PhD
at the
University of St. Andrews**



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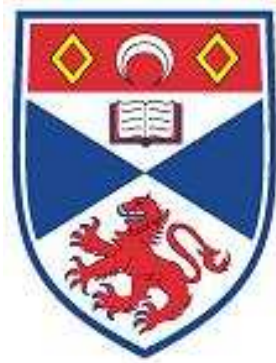
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**APPLICATION OF STOCHASTIC DIFFERENTIAL GAMES
AND REAL OPTION THEORY
IN ENVIRONMENTAL ECONOMICS**

Wen-Kai Wang

Submitted for the degree of
Doctor of Philosophy (Economics)
at the University of St Andrews

9 October 2009

Declarations

I, Wen-Kai Wang, hereby certify that this thesis, which is approximately 42,000 words in length, has been written by me, that it is the record of work carried out by me and that it has not been submitted in any previous application for a higher degree.

Date: 9 October 2009. Signature of Candidate:

I was admitted as a research student in September 2007 (credited with one year of study from the University of Leeds) and as a candidate for the degree of PhD Economics in September 2008; the higher study for which this is a record was carried out in the University of St Andrews between 2007 and 2009.

Date: 9 October 2009. Signature of Candidate:

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Abstract

This thesis presents several problems based on papers written jointly by the author and Dr. Christian-Oliver Ewald.

Firstly, the author extends the model presented by Fershtman and Nitzan (1991), which studies a deterministic differential public good game. Two types of volatility are considered. In the first case the volatility of the diffusion term is dependent on the current level of public good, while in the second case the volatility is dependent on the current rate of public good provision by the agents. The result in the latter case is qualitatively different from the first one. These results are discussed in detail, along with numerical examples.

Secondly, two existing lines of research in game theoretic studies of fisheries are combined and extended. The first line of research is the inclusion of the aspect of predation and the consideration of multi-species fisheries within classical game theoretic fishery models. The second line of research includes continuous time and uncertainty. This thesis considers a two species fishery game and compares the results of this with several cases.

Thirdly, a model of a fishery is developed in which the dynamic of the unharvested fish population is given by the stochastic logistic growth equation and it is assumed that the fishery harvests the fish population following a constant effort strategy. Explicit formulas for optimal fishing effort are derived in problems considered and the effects of uncertainty, risk aversion and mean reversion speed on fishing efforts are investigated.

Fourthly, a Dixit and Pindyck type irreversible investment problem in continuous time is solved, using the assumption that the project value follows a Cox-Ingersoll-Ross process. This solution differs from the two classical cases of geometric Brownian motion and geometric mean reversion and these differences are examined. The

aim is to find the optimal stopping time, which can be applied to the problem of extracting resources.

Keywords: Differential Games, Real Options, Stochastic Optimal Control, Public Goods, Fisheries, Maximum Sustainable Yields, Cox-Ingersoll-Ross Process, Environmental Economics

JEL Subject Classification: C61, C73, D21, D62, G11, G12, G31, Q22, Q57

Contents

Chapter 1

Introduction

Hanley, Sorgren and White (2007) have pointed out "A market failure occurs when the market does not allocate scarce resources to generate the greatest social welfare. A wedge exists between what a private person does given market prices and what society might want him or her to do to protect the environment. Such a wedge implies wastefulness or economic inefficiency; resources can be reallocated to make at least one person better off without making anyone else worse off.", see [?], page 42. Market failure may be caused by producing public goods. The main reason is that private sector producers will not supply public goods to people because they cannot be sure of making an economic profit. The notion of a public good was defined by Samuelson in his seminal 1954 paper "Collective consumption goods" [?]. According to this a public good is a good which is non-rival and non-excludable. That is, any consumer's consumption does not lead to a reduction in the amount of the good and each individual is allowed to take advantage of it. Some examples are academic research and national reputation. The principal question for an economic agent is, how much should he contribute to the public good? Potentially consumers can take advantage of the public good without sufficient contribution. This issue is called

the free rider problem. In environmental economics, climate change is a crucial global public good and this represents one classical example of a market failure, which leads all countries to benefit or suffer from it. Scientists have identified that the Earth has warmed 0.5°C over the past 100 years and green-house gas (GHG) concentrations have also increased significantly over the past 200 years. The United Nations Intergovernmental Panel on Climate Change (IPCC) has pointed out "the balance of evidence suggests that there is a discernible human influence on global climate.", see [?] Technical Summary E. Many scientists are concerned about reducing global GHG emissions in order to reduce the risk to human beings and the environment. Climate change could be caused by GHG emissions and it is indeed a threat to economic and ecological sustainability. How does climate change affect the economy?

One example is renewable resources. Fish stocks, a globally important renewable resource, have been affected by it. The ocean represents the main environmental heat sink and scientists have noticed that since 1961, around 80 per cent of the heat added to the climate system has been absorbed by the sea. This leads to an increase in sea temperature. The UN News Centre reports that the distribution of marine and freshwater fish has been affected by climate change and that biological processes are also influenced, see [?]. These impacts affect the biomass of species and therefore affect the harvest rates of fishery agents. GHG emissions accumulate and remain in the atmosphere for several hundred years. A reduction in GHG emissions may not cause the GHG concentrations to dissipate because they will take time to decay. The risks to human beings and the environment are caused by the aggregate stock of GHG and different rates of emissions may lead to the same concentrations. Therefore, policymakers should focus on how to achieve a given level of concentration. The Kyoto Protocol is a protocol to the United Nations Framework

Convention on Climate Change (UNFCCC), with the objective of stabilizing GHG emissions in the atmosphere at a level that would prevent dangerous anthropogenic interference with the climate system. However, developing countries such as China and India will soon be the largest emitters in the world due to their great demand for fossil fuels. Even though their intent to comply has been reported, "EU presses China and India to reduce greenhouse gas emissions", see [?], the protocol is self-enforcing and this leads to inefficiency. Climate protection is a public good and some countries may free ride off the efforts of others and this leads to a public good game.

As mentioned earlier, there are some environmental resources such as fisheries that are suffering from climate change. Cod has been an important economic commodity since the Viking period and some researchers indicate that the US Atlantic cod population may drop by as much as 50 percent by 2050 due to climate change, see [?]. In addition to this, such a severe impact will also affect other species because of the interactions between species. There are various reasons why multi species models should be studied. The general idea that individuals and countries should adopt an ecosystem approach to the sustainable use of natural resources in fact underpins many of the resolutions passed by the 2002 World Summit on Sustainable Development (WSSD) in Johannesburg. In particular, the WSSD plan of implementation requires signatory nations to develop and implement an ecosystem approach to fisheries (EAF) by 2012. From the point of view of conservation ecology, it is important to understand that the fishing of one species may have significant effects on another species and that this must be taken into account when thinking about the conservation of this species. There is no doubt that ecosystems are affected by uncertainty and interaction among species. It is impossible to operate a fishery without affecting the ecological equilibrium. In

order to ensure the conservation of the target species, fishery management measures should consider other species which belong to the same ecosystem or are associated with the target. Sharks are a slow-growing and long-lived species and they produce few offspring. They are a common seafood around the world in countries such as Japan and Australia and they are often killed for shark fin soup. Since they are apex predators and the population is comparatively small, a decrease in the stock of sharks leads the stocks of other species to increase or decrease significantly. On the other hand, to avoid the extinction of species, it is necessary to consider the behaviour among fishery agents. Each agent's harvest rate will affect other agents' decisions and this may cause species to become extinct. In order to understand the impact of fishery agents' behaviour on species, it is necessary to study a fishery game.

Even though people could benefit from a public good, they may suffer when the 'benefit' becomes too great. Tourism is one of the major contributions to the GDP of countries such as Greece and Spain. Both local residents and governments enjoy the economy improvements caused by tourists and may promote these holiday destinations to attract more holidaymakers. In other words, the residents and government benefit from the reputation of these destinations and the reputation represents a public good. However, a larger value of such good, i.e., a higher reputation, will be associated with a higher demand for facilities such as tourist accommodation and environmental problems such as pollution, loss of biodiversity and resource depletions are caused by a larger number of tourists and consequent over-development. The Mediterranean coastal zone, one of the invaluable assets around the world due to its biodiversity and cultural heritage, has suffered from over-development. Evidence of this has been identified in [?], "These demographic and tourist trends result in highly increasing infrastructures and facilities on the

coastal zone. As regards transport, intensively used roads now run along a large part of the Mediterranean coast at no more than a kilometre from the shoreline. Often constructed too close to the shores, the roads disrupt the physical exchanges between land and sea and generate a linear urbanisation along the coast. Certain airports, built on wetlands, contribute to the disappearance of ecosystems of great ecological and economic value”.

Another key topic in environmental economics is non renewable resources because the consumption of these resources permanently reduces the available stock. The extraction of such resources may be influenced by several factors, e.g., the price in the market. Such decisions can be regarded as irreversible investments, since they are related to big capitals and therefore the real option theory can be applied to investigate the problem of extracting non renewable resources. The main philosophy of the real options approach is that a financial manager faces the decision of and when to invest in a financial project. As opposed to the well known theory of options on stocks, it is generally assumed that the assets underlying real options are not traded on relatively liquid markets and, furthermore, that investment decisions are generally irreversible. Problems that fall into this category range from investment in real estate to problems of environmental economics and the reduction of GHG emissions, see Dixit (2000) [?]. Resources represent another typical example of real options. For example, oil could be considered as an option to invest in the development of a reserve, see [?], chapter 12. Agents decide when they should extract the oil due to the price of oil in the market. Real option theory can also be applied to the problem of renewable resources. Li (1998) studied a model in a fishery where he proposes that the stock of fish follows a geometric Brownian motion and this affects the agents’ decision about when and how much fish they are going to catch, see [?].

The motivation for this thesis is that environmental economics is becoming more and more popular and, because the stock of resources are limited, it will be necessary to introduce management policies. In the real world, time is continuous and it would be interesting to consider a continuous model. Two main techniques have been adopted in this thesis are: the real option approach and differential game theory, and these will be introduced in the following subsections. The following chapters are based on papers written jointly with the author's supervisor, Dr. Christian-Oliver Ewald, and they are available at SSRN. Two applications of differential game theory are discussed in Chapters 2 and 3, these are public goods and fishery games. In Chapter 2, it is considered that the public good satisfies the premise that an individual's consumption may be reduced if the value of the good exceeds a given level. One example of such a public good is conservation, because slow development of an economy may result from over-conservation of environmental resources. Chapter 3 examines a two species ecological system in which there are interactions between the two species. The analysis investigates how the ecological interactions affect fishery agents' decisions and how these decisions influence the ecological system. In order to combine climate change and ecological interactions among species, a stochastic model is considered where climate change represents one of reasons for the uncertainty. Following on from the numerical example in Chapter 3, an extension of the Gordon-Schaefer model is presented in Chapter 4 to provide a policy for fishery management. In Chapter 5, the proposition that a financial project follows a Cox-Ingersoll-Ross process is discussed.

1.1. Real option theory

Real option theory originated with the work of Myers (1977) [?] and is becoming more and more popular. There is no doubt that time always plays a crucial role

when a firm makes its investment decision. The decisions a firm makes in the present are affected by uncertainty and also by decisions made by other firms later on. To make the decision, the firm has to look ahead to all future possibilities and decide, whether the firm's investment should be postponed or not. To introduce real option theory, this section will first give a brief introduction to two techniques, dynamic programming and contingent claims analysis, and look at how they can be applied to identify the optimal stopping time.

Dynamic programming has been widely used in dynamic optimization, particularly when dealing with uncertainty. Starting with the simplest case, a two-period example, suppose that $I \geq 0$ is the sunk cost at each period and $\rho \in (0, 1)$ is a discount rate given exogenously. An agent faces the situation where he can determine whether he should invest at $t = 0$ or wait until $t = 1$. Suppose that the value of the project at $t = 0$ is given by P_0 and at $t = 1$, it is

$$P_1 = \begin{cases} (1 + u) P_0, & \text{with probability } p \\ (1 - d) P_0, & \text{with probability } 1 - p \end{cases},$$

Suppose that the agent has decided to invest in the project at $t = 0$ and let V_0 be the present value of the revenues he receives. The value V_0 discounted back to $t = 0$ can be derived by

$$\begin{aligned} V_0 &= P_0 + [p(1 + u) + (1 - p)(1 - d)] P_0 \sum_{i=1}^{\infty} \frac{1}{(1 + \rho)^i} \\ &= \frac{[1 + \rho + p(u + d) - d]}{\rho} P_0 \end{aligned}$$

and the payoff for the agent is defined by $\max\{V_0 - I, 0\}$. It can be seen that the agent invests only if V_0 is greater than the sunk cost I . On the other hand, if the agent does not invest at $t = 0$ but $t = 1$, the present value of the revenues discounted

back to $t = 1$ is given by

$$V_1 = P_1 \sum_{i=0}^{\infty} \frac{1}{(1+\rho)^i} = \frac{(1+\rho)}{\rho} P_1.$$

Similarly, the agent will not invest if V_1 is less than I and therefore, his payoff is given by $F_1 = \max \{V_1 - I, 0\}$. However, from the perspective of $t = 0$, the value of the project at $t = 1$ is stochastic, which leads V_1 to be stochastic as well. To determine whether it is worth investing at $t = 0$, the expectation of F_1 is taken subject to all information available at $t = 0$, which is given by

$$E \{F_1\} = p \max \left\{ \frac{(1+\rho)(1+u)P_0}{\rho} - I, 0 \right\} + (1-p) \max \left\{ \frac{(1+\rho)(1-d)P_0}{\rho} - I, 0 \right\}.$$

$E \{F\}$ is called the expected continuation value, or continuation value. Since the agent tries to maximize his payoff, the optimal decision can be made by comparing $\max \{V_0 - I, 0\}$ with $\frac{E\{F_1\}}{1+\rho}$ and his payoff is defined by

$$F_0 = \max \left\{ V_0 - I, \frac{E \{F_1\}}{1+\rho} \right\}.$$

In the two-period case, the agent could determine whether he should invest in the project at $t = 0$. If he decides not to invest at $t = 0$, then he has to wait until $t = 1$. In other words, his control, u_t at each t , can be represented by $u_t = 0$ for waiting and $u_t = 1$ for investing. This analysis can be extended for the case where the number of time periods is more than two. A more general situation can also be considered where the control that the agent has could be a continuous variable, for example, the amount of investment. It is assumed that there exists some states which are given by Markov processes; the definition of a Markov process is given below:

Definition 1. (*Markov process.*) A random process x_t is called a Markov process if it satisfies the Markov property, i.e., the conditional probability distribution of future states x_t of the process, given the history of the process up to and including time s , is dependent only on the state process x_s and independent of states before time s .

Definition 1 indicates that, to predict what will happen at $t + 1$, the information needed is given at t , not before t . Since the goal is to consider the uncertainty determined by a Wiener process and Wiener processes satisfy the Markov property, it is appropriate to give the assumption that states are Markov processes. It should first be considered that T is the terminal time, which is finite, and the termination payoff for the agent is defined by $\Omega_T(x_T)$, where x_t is the vector of states. Suppose that the current date is t and the agent has already chosen his control u_t , which leads him to immediately obtain profit flow $\pi(x_t, u_t)$. On the other hand, the state at $t + 1$ is influenced by u_t and x_t . Therefore, the continuation value is given by $E_t \{F_{t+1}(x_{t+1})\}$. Hence, to maximize the payoff at t , it is necessary to solve

$$F_t(x_t) = \max_{u_t} \left\{ \pi(x_t, u_t) + \frac{E_t \{F_{t+1}(x_{t+1})\}}{1 - \rho} \right\}. \quad (1.1)$$

Equation (1.1) is called the Bellman equation, which is named after its discoverer, Richard Bellman. In contrast with the two-period example, immediate investment implies that $\pi(x_0, 1) = V_0 - I$, while the agent gets only discount continuation value if he decides to wait, i.e., $\pi(x_0, 0) = 0$. This can be regarded as a special case of equation (1). In order to derive the optimal control at each period, it is possible to take advantage of the termination payoff $\Omega_T(x_T)$ and apply the backward method.

Since $\Omega_T(x_T)$ is known, it can be seen that the payoff at $t - 1$,

$$\begin{aligned} F_{T-1}(x_{T-1}) &= \max_{u_{T-1}} \left\{ \pi(x_{T-1}, u_{T-1}) + \frac{E_{T-1} \{F_T(x_T)\}}{1 - \rho} \right\} \\ &= \max_{u_{T-1}} \left\{ \pi(x_{T-1}, u_{T-1}) + \frac{E_{T-1} \{\Omega_T(x_T)\}}{1 - \rho} \right\} \end{aligned}$$

can be computed and therefore, the payoff at each t can be derived. However, in the case where T is infinite, the backward method collapses because there is no terminal time. This case can be solved by using a method similar to a two-period model. The stages at t and $t + 1$ are considered and the idea discussed above is applied. On the other hand, if the state x follows a difference equation which is independent of t explicitly and the profit flow $\pi(x, u)$ is also independent on t explicitly, the model is said to be autonomous; in other words, it is time homogeneous. Therefore, equation (1.1) can be written as

$$F(x) = \max_u \left\{ \pi(x, u) + \frac{E \{F(x') | x, u\}}{1 - \rho} \right\}, \quad (1.2)$$

where x is the current state and x' is the future state. The expectation $E \{F(x') | x, u\}$ is a conditional expectation where the information is given by the current state x and control u . Such an assumption always appears in the case of infinite time horizon models and it is reasonable that, in this case, one makes decisions based on the state, not the time. Numerically, to find the optimal control, first take any guess $F^1(x)$ and find the corresponding u^1 , which is a function of x . Substituting this into equation (1.2), give another function, say $F^2(x)$. Repeating this procedure generates a sequence of functions, $\{F^1(x), F^2(x), \dots, F^i(x), \dots\}$. It can be shown that this sequence is convergent and the proof of this can be found in [?], Chapters 4 and 9.

So far, this analysis is only valid for a discrete model. The goal of this analysis is to consider the situation where the model is continuous and uncertainty is given by some Wiener processes. Therefore, it is necessary to extend the discussion to continuous time. Supposing that the length of each period is Δt instead of 1. The rate for profit flow $\pi(t, x, u)$ over each interval of time is defined by $\pi(t, x, u)\Delta t$ and the discount continuation value is given by $\frac{E\{F(t+\Delta t, x')|x, u\}}{1+\rho\Delta t}$. Therefore, equation (1.2) becomes

$$F(t, x) = \max_u \left\{ \pi(t, x, u)\Delta t + \frac{E\{F(t + \Delta t, x')|x, u\}}{1 + \rho\Delta t} \right\},$$

which then implies

$$F(t, x) (1 + \rho\Delta t) = \max_u \{ \pi(t, x, u)\Delta t (1 + \rho\Delta t) + E\{F(t + \Delta t, x')|x, u\} \},$$

or

$$\begin{aligned} \rho\Delta t F(t, x) &= \max_u \{ \pi(t, x, u)\Delta t (1 + \rho\Delta t) + E\{F(t + \Delta t, x') - F(t, x)|x, u\} \} \\ &= \max_u \{ \pi(t, x, u)\Delta t (1 + \rho\Delta t) + E\{\Delta F\} \}. \end{aligned}$$

Divide both sides in the above equation by Δt and let Δt tend to 0, then

$$\rho F(t, x) = \max_u \left\{ \pi(t, x, u) + \frac{E\{dF\}}{dt} \right\}. \quad (1.3)$$

Now, assuming that the value of the financial project is given by a stochastic differential equation:

$$dx(t) = a(t, x, u)dt + b(t, x, u)dW(t),$$

where $W(t)$ is a Wiener process. Application of Itô formula gives:

$$dF(t, x(t)) = \left(\frac{\partial}{\partial t} F(t, x(t)) + a(t, x(t), u(t)) \frac{\partial}{\partial x} F(t, x(t)) + \frac{b^2(t, x(t), u(t))}{2} \frac{\partial^2}{\partial x^2} F(t, x(t)) \right) dt + b(t, x(t), u(t)) \frac{\partial}{\partial x} F(t, x(t)) dW(t).$$

Since $E \{ b(t, x(t), u(t)) \frac{\partial}{\partial x} F(t, x(t)) dW(t) \} = 0$, $\frac{E\{dF\}}{dt}$, can be represented by

$$\frac{E\{dF\}}{dt} = \frac{\partial}{\partial t} F(t, x(t)) + a(t, x(t), u(t)) \frac{\partial}{\partial x} F(t, x(t)) + \frac{b^2(t, x(t), u(t))}{2} \frac{\partial^2}{\partial x^2} F(t, x(t)),$$

and the Bellman equation is defined by

$$\rho F(t, x) = \max_u \left\{ \pi(t, x, u) + \frac{\partial}{\partial t} F(t, x) + a(t, x, u) \frac{\partial}{\partial x} F(t, x) + \frac{b^2(t, x, u)}{2} \frac{\partial^2}{\partial x^2} F(t, x) \right\}. \quad (1.4)$$

Therefore, the Bellman equation is already derived for the case where the project value given by an Itô process.

Suppose that the agent can determine whether he should invest at t . If the investment is irreversible, then this raises a question, i.e., what is the optimal timing for investing? This is called optimal stopping time. In a two-period model, the optimal stopping is determined by $F_0 = \max \left\{ V_0 - I, \frac{E\{F_1\}}{1+\rho} \right\}$. If $F_0 = V_0 - I$, then this means that the agent invests at $t = 0$. On the other hand, if $F_0 = \frac{E\{F_1\}}{1+\rho}$, then he waits until $t = 1$ and then invests. It can be extended to a model with many periods. If the termination payoff is defined by $\Omega(t, x)$, then the Bellman equation becomes

$$F(t, x) = \max \left\{ \Omega(t, x), \pi(t, x) + \frac{E\{F(t + \Delta t, x') | t, x\}}{1 + \rho} \right\}. \quad (1.5)$$

Similarly, the agent would not invest if $F(t, x) = \pi(t, x) + \frac{E\{F(t + \Delta t, x') | t, x\}}{1 + r}$. Now,

moving to the case where the financial project is defined by an Itô process. Equation (1.4) implies that

$$\rho F(t, x) = \pi(t, x) + \frac{\partial}{\partial t} F(t, x) + a(t, x) \frac{\partial}{\partial x} F(t, x) + \frac{b^2(t, x)}{2} \frac{\partial^2}{\partial x^2} F(t, x). \quad (1.6)$$

To determine the optimal stopping, note that, in a discrete case, the termination payoff and expected continuation value are compared. In a continuous model, it is reasonable that the optimal stopping satisfies the condition $F(t, x^*(t)) = \Omega(t, x^*(t))$, which is called the value-matching condition. Moreover, this is a free-boundary problem because x^* is unknown. Mathematically, this is always more complicated than an initial value problem. On the other hand, there is an additional condition which comes from economic consideration and this is defined by $\frac{\partial}{\partial x} F(t, x^*) = \frac{\partial}{\partial x} \Omega(t, x^*)$. This condition is named the smooth-pasting condition. The value-matching and smooth-pasting conditions do indeed give the optimal stopping. The value-matching condition requires that at the time the agent invests, the payoff and the continuation value subtracting the sunk cost are equal. On the other hand, the smooth-pasting condition guarantees the continuity of the first derivative at x^* . To see how both conditions work, an example can be found in [?], see page 130, Appendix C. On the other hand, if the time horizon is infinity, and $\pi(x)$, $a(x)$, $b(x)$ and $\Omega(x)$ are not explicitly dependent on time, then the equation for the model is autonomous and equation (1.6) can be rewritten as

$$\rho F(x) = \pi(x) + a(x) F'(x) + \frac{b^2(x)}{2} F''(x). \quad (1.7)$$

with two conditions, $F(x^*) = \Omega(x^*)$ and $F'(x^*) = \Omega'(x^*)$. Comparing equation (1.6) with (1.7), it can be seen that, in the case with a finite time horizon, the optimal stopping is not only dependent on time, but also the state, which is the value of

the project. In the case where the time horizon is infinite, the agent chooses the optimal stopping according to the state x . This is realistic because he can wait until the continuation value reaches the termination payoff without facing any terminal date.

This section now moves on to another technique, named contingent claims analysis. In dynamic programming, $F(t, x)$ is interpreted as the value of an asset in the market, while in contingent claim analysis, $F(t, x)$ could be regarded as the output of a firm, e.g., oil and copper production. This can be traded as an asset in financial markets. The discount rate ρ in dynamic programming is given exogenously, and this may not be easy to specify. On the other hand, in contingent claims analysis, it is assumed that the traded asset has a risk adjusted expected rate of return μ and therefore the firm pays a risk premium of $\mu - r$, where r is the riskless rate of return given exogenously. Suppose that the financial project $x(t)$ follows the stochastic differential equation

$$dx(t) = \alpha(t, x(t))dt + \sigma(t, x(t))dW(t),$$

where $W(t)$ is a standard Wiener process. The expected rate of return of the investment is given by $\frac{\alpha(t, x(t))}{x(t)}$ and for arbitrage reasons it would need to pay a dividend rate of

$$\delta(t, x(t)) = \mu - \frac{\alpha(t, x(t))}{x(t)}.$$

In the context of real options this rate is called the implied proportional dividend rate. Suppose that an agent invests one dollar and buys n units of x . The agent holds this for an interval of time dt and obtains $r dt$ from the riskless asset. On the other hand, the other asset pays a dividend $n\delta(t, x(t))x(t)dt$ and has a random capital gain, $ndx(t) = n\alpha(t, x(t))dt + n\sigma(t, x(t))dW(t)$. Therefore, the total return

on each dollar invested is

$$\frac{r + n(\alpha(t, x(t)) + \delta(t, x(t))x(t))}{1 + nx(t)}dt + \frac{n\sigma(t, x(t))}{1 + nx(t)}dW(t).$$

On the other side, if the output of the firm is given by $F(t, x(t))$ and the profit flow is defined by $\pi(t, x(t))$, according to the Itô's formula:

$$\begin{aligned} dF(t, x(t)) = & \left(\frac{\partial}{\partial t}F(t, x(t)) + \alpha(t, x(t))\frac{\partial}{\partial x}F(t, x(t)) + \frac{\sigma^2(t, x(t))}{2}\frac{\partial^2}{\partial x^2}F(t, x(t)) \right) dt \\ & + \sigma(t, x(t))\frac{\partial}{\partial x}F(t, x(t))dW(t). \end{aligned}$$

If the portfolio can replicate the risk of owning the firm, this gives the following equations:

$$\begin{aligned} \frac{\pi(t, x) + \frac{\partial}{\partial t}F(t, x) + \alpha(t, x)\frac{\partial}{\partial x}F(t, x) + \frac{\sigma^2(t, x)}{2}\frac{\partial^2}{\partial x^2}F(t, x)}{F(t, x)} &= \frac{r + n(\alpha(t, x) + \delta(t, x)x)}{1 + nx}, \\ \frac{\sigma(t, x)\frac{\partial}{\partial x}F(t, x)}{F(t, x)} &= \frac{n\sigma(t, x)}{1 + nx} \end{aligned}$$

Therefore,

$$\frac{\sigma^2(t, x)}{2}\frac{\partial^2}{\partial x^2}F(t, x) + (r - \delta(t, x))x\frac{\partial}{\partial x}F(t, x) + \frac{\partial}{\partial t}F(t, x) - rF(t, x) + \pi(t, x) = 0, \quad (1.8)$$

with the conditions $F(t, x^*) = \Omega(t, x^*)$ and $\frac{\partial}{\partial x}F(t, x^*) = \frac{\partial}{\partial x}\Omega(t, x^*)$. Note that if the model is autonomous, equation (1.8) becomes:

$$\frac{\sigma^2(x)}{2}F''(x) + (r - \delta(x))xF'(x) - rF(x) + \pi(x) = 0. \quad (1.9)$$

with the conditions $F(x^*) = \Omega(x^*)$ and $F'(x^*) = \Omega'(x^*)$.

It can be seen that equations (1.6) and (1.8) are of a similar structure, r replaces

ρ and $(r - \delta(t, x))x$ replaces $a(t, x)$. The interpretation of the contingent claims analysis differs from that in dynamic programming: In dynamic programming, the agent determines whether the asset is worth holding, while in contingent claims analysis, the agent chooses the option exercise date in order to maximize the value of the asset. When working with dynamic programming, a discount rate is required, thus ρ is given exogenously, while in contingent claim analysis, the riskless rate of return r is needed and this is easy to specify compared to ρ . However, in contingent claim analysis, a rich set of markets in risky assets is necessary because the intension is to replicate the uncertainty $W(t)$ by some risky assets in markets.

1.2. Stochastic differential games

Differential game theory was introduced by Rufus Isaacs in 1965, see [?]. In classical game theory, players are allowed to make decisions at a particular time. However, in the real world, time is continuous and this assumption may not be realistic. Differential game theory differs from classical game theory in some aspects: All players face a continuous time horizon and they are allowed to make their decisions at any time. Each player tries to maximize his objective functional, which can be interpreted as the payoff. Moreover, payoffs are accumulated over time. All players face a family of states defined by differential equations. The interpretation of these states is dependent on the problem that is being dealt with. For example, in a public good game, the state is interpreted as the value of the public good, whereas in a fishery game, states are understood to be the biomass of species. These cases will be introduced in Chapters 2 and 3, respectively. The techniques that are applied to deal with differential games are the Hamilton-Jacobi-Bellman equation and Pontryagin's maximum principle. A deterministic differential game

model is always given, for each player i ,

$$\max_{u_i} \int_{t_0}^T e^{-r_i(t-t_0)} F_i(t, x(t), u_i(t), u_{-i}(t)) dt + e^{-r(T-t_0)} S_i(x(T)), \quad (1.10)$$

subject to

$$dx(t) = f(t, x(t), u_i(t), u_{-i}(t)) dt, \quad x(t_0) = x_0, \quad (1.11)$$

where

$$u_{-i}(t) = (u_1(t), \dots, u_{i-1}(t), u_{i+1}(t), \dots, u_N(t))$$

is the strategies chosen by other players. u_i is the vector of controls for player i and r_i is a discount rate. $S_i(x)$ is called the transversality condition, which is interpreted as the terminal payoffs of player i . To solve the system (1.10) and (1.11), the Hamilton-Jacobi-Bellman equation and Pontryagin's maximum principle are introduced. There are defined below:

Theorem 1. (Hamilton-Jacobi-Bellman equation.) *Let $V : X \times [0, T] \rightarrow \mathbb{R}$ be a continuously differential function, where X is the state space. If $V(t, x)$ satisfies the Hamilton-Jacobi-Bellman equation*

$$rV(t, x) - \frac{\partial}{\partial t} V(t, x) = \max_u \left\{ F(t, x, u) + \frac{\partial}{\partial x} V(t, x) f(t, x, u) \right\},$$

with the terminal condition $V(T, x) = S(x)$, then u^ maximizing the right hand side of the Hamilton-Jacobi-Bellman equation is an optimal control and $V(t, x)$ is the optimal value.*

where $\frac{\partial}{\partial x} V(t, x)$ is denoted by the gradient of $V(t, x)$ with respect to x , and

Theorem 2. (Pontryagin's maximum principle.) *Let*

$$H(t, x(t), \lambda(t), u(t)) = F(t, x(t), u(t)) + \lambda(t)f(t, x(t), u(t))$$

be the Hamiltonian function and suppose that

$$H^*(t, x(t), \lambda(t)) = \max_u \{H(t, x(t), \lambda(t), u(t))\}$$

exists. Assume that X , the state space, is convex and the transversality condition $S(x) \in C^1$, where C^1 is the class of all continuously differentiable functions, is convex. If there exists an absolutely continuous function $\lambda(t)$ satisfying $H^(t, x(t), \lambda(t)) = \max_u \{H(t, x(t), \lambda(t), u(t))\}$, the costate equation $\lambda'(t) = r\lambda(t) - \frac{\partial}{\partial x} H^*(t, x(t), \lambda(t))$ and $\lambda(T) = S'(x(T))$ as well as $H^*(t, x, \lambda(t)) \in C^1$ is concave with respect to x , then u^* which maximizes the Hamiltonian function is an optimal control.*

Theorem 1 and 2 are versions of a finite time horizon, and to solve a deterministic differential game model, Theorem 1 or 2 are applied to derive the value function $V_i(t, x)$ for each player i and then solve a system of partial differential equations. Each partial differential equation has an associated boundary condition $V_i(T, x) = S_i(x)$. However, in the real world, some situations may be faced where the terminal time is unknown or the time duration is infinity, i.e., $T = \infty$. In this case, the transversality condition is usually given by $S_i(x) = 0$ because there is no terminal date and terminal payoff. Theorem 1 and 2 can be extended to versions capable of resolving infinite time horizon. The idea is to give a finite terminal time T and then derive u^* and $V(t, x; T)$. If $\lim_{T \rightarrow \infty} V(t, x; T)$ exists and this solves the Hamilton-Jacobi-Bellman equation for the infinite time horizon model, then $V(t, x)$ is the optimal value function. This idea is called the finite horizon approximation approach, which is mentioned in [?], see page 70. The finite horizon approximation

approach is used because, with an infinite time horizon, the terminal condition for the Hamilton-Jacobi-Bellman equation is given by $\lim_{T \rightarrow \infty} V(T, x; T) = 0$, which is not helpful when solving the partial differential equation of $V(t, x)$. On the other hand, it is possible to guess the form of solutions and then derive the unknown coefficients. However, it may not be easy to pick the correct values. One example of this is given in the paper by Ewald and Wang (2009), see [?]. This is based on a stochastic model, where it can easily be supposed that the volatility is 0. Moreover, the solution, derived by finite horizon approximation approach, satisfies the catching-up optimality, i.e.,

$$\liminf_{T \rightarrow \infty} V(t, u^*(t); T) - V(t, u(t); T) \geq 0,$$

where u^* is an optimal control for the finite horizon model with terminal time T and u is any feasible control. Let $[0, T\rangle$ be denoted by the time duration that players face. There are three types of Nash-equilibrium strategies: open-loop Nash-equilibrium strategies, feedback Nash-equilibrium strategies and stationary feedback Nash-equilibrium strategies. Feedback Nash-equilibrium strategies are sometimes called closed-loop Nash-equilibrium strategies or Markovian Nash-equilibrium strategies. Each type has its own interpretation. These are the definitions for each type:

Definition 2. (*Open-loop Nash equilibrium.*) The N-tuple (u_1^*, \dots, u_N^*) of functions $u_i^* : [0, T\rangle \rightarrow \mathbb{R}^{m_i}$, is called an open-loop Nash equilibrium if u_i^* solves the system (1.10) and (1.11) and it is a function of time.

Definition 3. (*Feedback Nash equilibrium.*) The N-tuple (u_1^*, \dots, u_N^*) of functions $u_i^* : X \times [0, T\rangle \rightarrow \mathbb{R}^{m_i}$ where X is the state space, is called a feedback Nash equilibrium if u_i^* solves the system (1.8) and (1.9) and is a function of both time

and states.

Definition 4. (*Stationary feedback Nash equilibrium.*) The N-tuple (u_1^*, \dots, u_N^*) of functions $u_i^* : X \rightarrow \mathfrak{R}^{m_i}$, is called a stationary feedback Nash equilibrium if u_i^* solves the system (1.8) and (1.9) and it is a function of states.

Definition 2 says that an open-loop Nash-equilibrium strategy is dependent on time. It always appears in a finite time horizon model because each player knows the terminal time and time definitely affects their strategies. For example, in a public good game, due to the free rider effect, it is intuitive that each player has less incentives to invest until the terminal date is approaching. On the other hand, when a player decides to employ a feedback Nash-equilibrium strategy, he is concerned not only with current time, but also current states. This could be used in a situation where some players know that the states they are currently facing may not be correct. Because all players derive their optimal strategies from the states, those players who realise this can take advantage of the correct current states and then make their decisions. Definition 4 always occurs when players are facing an infinite time horizon. Mathematically, if the model is autonomous, one could derive a stationary feedback Nash-equilibrium strategy if such a strategy exists. In the case with an infinite time horizon, open-loop Nash equilibria and stationary feedback Nash equilibria represent different phenomena. Taking the deterministic public good game in [?], section 9.5, as an example: If players decide to employ open-loop Nash equilibrium strategies, because there is no terminal date, it can be seen that the open-loop Nash equilibrium is convergent to a constant. This implies that the players have an expectation of what value the good will converge to. However, this ignores an important phenomenon, the free ride effect. On the other hand, when considering a stationary feedback Nash-equilibrium strategy, it can be seen that the free rider effect will appear. A larger value of the public good leads players to have

less incentive to invest because they will try to free ride on others. The case of the stochastic version will be introduced in Chapter 2.

It can be seen that, uncertainty always plays an important role in the real world. The introduction above only examines a deterministic model. One idea that could extend a deterministic differential game to a stochastic differential game is to consider that the uncertainty is determined by a Wiener process. In this case, the stochastic differential game is defined, for each player i ,

$$\max_{u_i} E \left\{ \int_{t_0}^T e^{-r_i(t-t_0)} F_i(t, x(t), u_i(t), u_{-i}) dt + e^{-r(T-t_0)} S_i(x(T)) \right\}, \quad (1.12)$$

subject to

$$dx(t) = f(t, x(t), u_i(t), u_{-i}(t))dt + \sigma(t, x(t), u_i(t), u_{-i}(t))dW(t), \quad x(t_0) = x_0, \quad (1.13)$$

where $\sigma(t, x(t), u_i(t), u_{-i}(t))$ is a matrix of functions and $W(t)$ is a vector of some independent Wiener processes. It can be seen that in equation (1.12), players maximize their expected utility since they face uncertainty. In contrast with a deterministic differential game, in a stochastic differential game, there are still three types of Nash-equilibrium strategies. However, the concept of open-loop Nash-equilibrium strategies may not be appropriate when facing a stochastic model. The reason for this is that uncertainty now appears in the model and it is harder to react by only considering time. Therefore, a feedback Nash-equilibrium strategy is always employed in a stochastic differential game. Another difference can be seen in the techniques used to solve a model. In a deterministic differential game, there are two methods, the Hamilton-Jacobi-Bellman equation and Pontryagin's maximum principle, used to solve a model. Nevertheless, it is very difficult to apply Pontryagin's maximum principle to deal with a stochastic differential game,

and the Hamilton-Jacobi-Bellman equation is widely used when facing a stochastic differential game. The stochastic version of the Hamilton-Jacobi-Bellman equation is given by

Theorem 3. (Hamilton-Jacobi-Bellman equation.) *Let $V : X \times [0, T] \rightarrow \mathfrak{R}$ be a function and $\frac{\partial}{\partial t}V(t, x)$, $\frac{\partial}{\partial x}V(t, x)$ and $\frac{\partial^2}{\partial x^2}V(t, x)$ are continuous. If $V(t, x)$ satisfies the Hamilton-Jacobi-Bellman equation*

$$rV(t, x) - \frac{\partial}{\partial t}V(t, x) = \max_u \left\{ F(t, x, u) + \frac{\partial}{\partial x}V(t, x)f(t, x, u) + \frac{1}{2}tr \left(\frac{\partial^2}{\partial x^2}V(t, x)\sigma(t, x, u)\sigma(t, x, u)' \right) \right\},$$

where $\sigma(t, x, u)$ is the matrix of volatility and $\sigma(t, x, u)'$ is the transpose of $\sigma(t, x, u)$, then u^* maximizing the right hand side of the Hamilton-Jacobi-Bellman equation is an optimal control and $V(t, x)$ is the expected optimal value.

Similarly to Theorem 1, Theorem 3 can be extended to the case with $T = \infty$. The concept is analogous to the deterministic version. In the deterministic version, the Hamilton-Jacobi-Bellman equation is given by a first order partial differential equation, while in the stochastic version, it is defined by a second order differential equation. The reason is due to Itô's formula and it can be seen that the volatility appears only in the coefficient for the term $\frac{\partial^2}{\partial x^2}V(t, x)$. The proofs for Theorem 1, 2 and 3 can be referred to [?] and they have been omitted here.

Chapter 2

Dynamic voluntary provision of public goods with uncertainty

Various papers have discussed the free rider problem within a static game theoretic model. Bergstrom, Blume and Varian (1986) [?] considered this case and proved that there exists a unique Nash equilibrium under very weak assumptions. Various other studies have considered the static case, see [?], [?], [?], [?] and [?]. In many contexts however a dynamic model where agents can adjust their provisions toward a public good depending on the current state of the system appears to be more reasonable and is therefore worthy of investigation. McMillan (1979) studied an infinitely repeated game and showed that the free rider problem may not be apparent if the value of public goods are not discounted too heavily, see [?]. McMillan's setup differs slightly from the line taken in many other studies in the way that he uses trigger strategies instead of continuous adjustment of the provision rate. However within this setup he does establish that the non-cooperative Nash-equilibrium is Pareto optimal, which is a remarkable result. This chapter will follow the approach taken by Fershtman and Nitzan (1991) who presented a continuous time model

with infinite time horizon with no uncertainty [?]. These authors suppose that the benefits and costs are accumulated over time and derived feedback Nash equilibria.

In many applications of public good theory, uncertainty plays a fundamental role, for example, with insurance. For this reason the setup of Fershtman and Nitzan has been extended to include an uncertainty term. Two different cases will be studied: in the first one the volatility of the uncertainty term depends exclusively on the current level of the public good. In this case the form of the feedback Nash-equilibrium is identical to the deterministic case. The level of the public good however fluctuates randomly. In the second case, the volatility of the uncertainty is dependent on the current rate of public good provision by the agents. This case results in a qualitatively different result. From an economic viewpoint, both scenarios are realistic. A large project value is associated with a higher risk. On the other hand, if an investor invests a great amount of money in a short amount of time, the level of the public good is generally also exposed to higher uncertainty. In reality there will be a mixture of both effects, but in this chapter they will be strictly separated for reasons of tractability and in order to highlight the differences.

This chapter will concentrate on the symmetric framework and the objective is to compute a symmetric feedback Nash-equilibrium in the sense of a stationary Markovian Nash equilibrium, which has been defined in section 1.2, Definition 4. The following section, 2.1, will give a brief introduction to the deterministic model that was studied by Fershtman and Nitzan (1991) [?]. This model will be extended by the introduction of a general uncertainty term in section 2.2. Section 2.3 will focus on the case where the volatility of the uncertainty exclusively depends on the current level of the public good, while section 2.4 examines, the case where it exclusively depends on the rate of contribution to the public good. Cooperative and non cooperative cases will be compared in both section 2.3 and 2.4. Various

numerical results are discussed in section 2.3 and 2.4, as well as presented in the form of figures in the end of this chapter. Some conclusions are made in section 2.5.

2.1. The deterministic model

This section will start with, a brief introduction of the deterministic model studied by Fershtman and Nitzan (1991) [?]. Here, the project value is given by the following controlled differential equation

$$x'(t) = \sum_{i=1}^n u_i(t) - \delta x(t), \quad x(0) = x_0. \quad (2.1)$$

the parameter δ is called a depreciation rate. It is assumed to be non-negative. The control $u_i(t)$ represents the amount of money invested into the project by investor i , while n is the number of investors. It would be expected that a higher value of depreciation rate would lead to a lower value of both the project and utility. Furthermore, investors may have less incentive to invest under a large depreciation rate. Each agent faces an individual cost given by $C(u_i(t))$ but they all benefit from the project in the same way. More precisely the benefit for each agent is given by $\alpha f(x(t))$, where α is greater than 0 and less than 1. Fershtman and Nitzan called the project a pure public good if $\alpha = 1$, otherwise it represents a combination of public and private good. Mathematically however, α does not play an important role here, since it can be assumed that $\tilde{f}(x(t)) = \alpha f(x(t))$ and replacement of \tilde{f} with f leads to $\alpha = 1$. Therefore, it is only necessary to concentrate on the case of $\alpha = 1$. For each agent i , the objective functional is defined by

$$\max_{u_i} \int_0^{\infty} e^{-rt} [f(x(t)) - C(u_i(t))] dt \quad (2.2)$$

subject to equation (2.1), where r is a discount rate. In order to obtain a mathematically tractable model Fershtman and Nitzan proposed that the cost for investor i is given by $C(u_i) = \frac{u_i^2}{2}$, and that the project value at t is given by $f(x) = ax - bx^2$, where $x < \frac{a}{2b}$, which therefore leads the model to be a linear quadratic game, see [?], Chapter 7. Note that under (2.1), the condition, $x < \frac{a}{2b}$, will always hold, providing that each investor chooses to invest according to the unique open loop Nash equilibrium. If the assumption $x < \frac{a}{2b}$ is relaxed, this can instead be thought of as a public good that satisfies the property that one suffers from it if its value is too large. One example indicated earlier in Chapter 1 is the over development of an environmental resource. To solve this model, Fershtman and Nitzan apply the Pontryagin maximum principle. As the model developed in this thesis includes uncertainty in the form of diffusion terms, the Hamilton-Jacobi-Bellman approach will be used. The following analysis will concentrate on the case of two investors, because simplifies the notation. The general case can be treated in analogy. As the focus will be on a symmetric Nash-equilibrium, the value function for both agents will be the same and $V_i(x)$ will be written as $V(x)$. Note that this model is autonomous and hence, according to the Hamilton-Jacobi-Bellman equation, if u_j^* is a stationary Nash feedback optimal control for agent j , this will give the differential equation

$$rV(x) = \max_{u_i} \left\{ -\frac{u_i^2}{2} + (ax - bx^2) + V'(x) (u_i + u_j^* - \delta x) \right\} \quad (2.3)$$

A necessary condition to maximize equation (2.3) is $u_i^* = V'(x)$, the marginal benefit of the value function. Substituting this into equation (2.3), shows that the value function is the solution of the ordinary differential equation:

$$\frac{3}{2} [V'(x)]^2 - \delta x V'(x) - rV(x) - bx^2 + ax = 0 \quad (2.4)$$

When considering the infinite horizon case, there is no terminal condition for the differential equation. Without further specification, there are an infinite number of solutions to equation (2.4). Instead of employing transversality conditions, which are more suitable for the Pontryagin maximum principle approach, this analysis will employ the technique of finite horizon approximation approach. If a sufficiently large expiry time T is given, the terminal condition is given by $V(T, x; T) = 0$ and this implies

$$\lim_{T \rightarrow \infty} V(T, x; T) = 0. \quad (2.5)$$

Assuming a functional form $V(t, x; T) = A(t; T)x^2 + B(t; T)x + C(t; T)$, equation (2.4) can be solved with respect to the terminal condition $V(T, x; T) = 0$. Substitution into the Hamilton-Jacobi-Bellman equation produces three differential equations for $A(t; T)$, $B(t; T)$ and $C(t; T)$. These are given by

$$\begin{aligned} A'(t; T) &= -6A^2(t; T) + (r + 2\delta)A(t; T) + b, \quad A(T; T) = 0 \\ B'(t; T) &= (r + \delta - 6A(t; T))B(t; T) - 1, \quad B(T; T) = 0 \\ C'(t; T) &= rC(t; T) - \frac{3}{2}B^2(t; T), \quad C(T; T) = 0 \end{aligned}$$

Their fixed points can be computed as $A = A^\pm$, $B_c = \frac{a}{r-6A+\delta}$ and $C_c = \frac{3B^2}{2r}$, where $A^\pm = \frac{(r+\delta) \pm \sqrt{(r+\delta)^2 + 24b}}{12}$. Since equation (2.5) holds, by analyzing the (A, A') -diagram, it can be seen that $A(t)$ must lie in $[A^-, 0]$ and its limit is $A_c = A^-$ for T tending to infinity. Therefore, the corresponding value function is given by $V(x) = A_c x^2 + B_c x + C_c$ and the stationary feedback Nash equilibrium can be computed as $u_i^* = 2A_c x + B_c$. The corresponding state equation is then given by

$$x'(t) = (4A_c - \delta)x(t) + 2B_c, \quad x(0) = x_0 \quad (2.6)$$

The solution can be computed and the limit is given by $x = \frac{2B_c}{\delta - 4A_c}$ as T tends to infinity. Note that a larger value of δ indeed implies a lower project value and lower optimal utility. On the other hand, the open-loop Nash-equilibrium strategy

$$\frac{4A_c B_c}{\delta - 4A_c} - \frac{2A_c}{\delta - 4A_c} (2B_c + 4A_c - \delta) e^{(4A_c - \delta)t} + B_c,$$

tends to $\frac{4A_c B_c}{\delta - 4A_c} + B_c > 0$ for t tending to infinity. As indicated in Chapter 1, it is harder to observe the free ride effect if agents adopt the open-loop Nash-equilibrium strategies and this is one disadvantage when studying a public good game model.

2.2. A stochastic version of the Fershtman and Nitzan model

The analysis in this section will extend the model discussed previously section by introducing uncertainty in the form of diffusion terms. This is one step toward a more realistic model, because in the real world agents face uncertainty. To construct the model, without loss of generality, it is supposed that there are only two agents and equation (2.1) can be extended to:

$$dx(t) = [u_1(t) + u_2(t) - \delta x(t)] dt + \sigma(u_1(t), u_2(t), x(t)) dW(t), \quad x(0) = x_0, \quad (2.7)$$

where $W(t)$ is a Wiener process. The volatility $\sigma(u_1(t), u_2(t), x(t))$ determines the level of uncertainty. The objective of agent i is given by

$$\max_{u_i} E \left\{ \int_0^\infty e^{-rt} [f(x(t)) - C(u_i(t))] dt \mid x(0) = x_0 \right\} \quad (2.8)$$

subject to equation (2.7). It is still assumed that individual costs are given by $C(u_i) = \frac{u_i^2}{2}$ and benefit functions are determined by $f(x) = ax - bx^2$, exactly as in Fershtman and Nitzan (1991) [?]. Note that this model is time homogeneous. To

solve it, the Hamilton-Jacobi-Bellman equation for the model is derived as

$$rV(x) = \max_{u_i} \left\{ -\frac{u_i^2}{2} + ax - bx^2 + \frac{\partial}{\partial x}V(x) (u_i + u_j^* - \delta x) + \frac{\sigma^2((u_1, u_2, x))}{2} \frac{\partial^2}{\partial x^2}V(x) \right\}, \quad (2.9)$$

where u_j^* is the optimal control for investor j . As before, interest is in a symmetric Nash-equilibrium and it can therefore be assumed that $u_1^* = u_2^*$. Equation (2.9) then enables the optimal control for agent i to be computed via

$$u_i^*(x) = \frac{\partial}{\partial x}V(x) + \sigma((t, u_1, u_2, x)) \frac{\partial}{\partial u_i} \sigma((u_1, u_2, x)) \frac{\partial^2}{\partial x^2}V(x). \quad (2.10)$$

From equation (2.10) it can be seen that, if the volatility function $\sigma((u_1(t), u_2(t), x(t)))$ does not explicitly depend on the controls, the optimal control in (2.10) is given by $u_i^*(x) = V'(x)$. This is formally the same as in the deterministic case, i.e. investment occurs according to marginal benefits from the public good. Note, however, that the value function changes due to the second order term in (2.9). As in the previous chapter, the technique of finite horizon approximation approach is applied to solve (2.9). In the general case where no analytical solution can be found, it is necessary to rely on numerical techniques so it is important to choose a sufficiently large T and then solve equation (3.9) with a suitable algorithm for a two dimensional partial differential equation, for example, the implicit method and Crank-Nicholson. Once a solution is obtained, all $V(t, x; T)$ terms can be examined by using the inequality:

$$\left| rV(t, x; T) + \frac{(u_i^*)^2}{2} - ax + bx^2 - V'(t, x; T) (u_i^* + u_j^* - \delta x) - \frac{\sigma^2(u_i^*, u_j^*, x)}{2} V''(t, x; T) \right| < \epsilon, \quad (2.11)$$

for all sufficiently large t and a given sufficiently small ϵ . Inequality (2.11) is equivalent to

$$\left| \frac{\partial}{\partial t} V(t, x; T) \right| < \epsilon'$$

for sufficiently small ϵ' . However, equation (2.9) is simulated only under a small interval of state x instead of a large one. The idea is that it is only necessary to have an initial condition and then algorithms for ordinary differential equations can be applied, for example, the Runge-Kutta method, to solve equation (2.9) numerically. Since equation (2.9) is only a second order ordinary differential equation, this will be more efficient than solving a partial differential equation. On the other hand, equation (2.9) can be solved numerically by the Markov chain approach, which was introduced in [?]. The concept used is to discretise (2.9) by the finite difference method and then determine the so called transition probabilities, which represent how the current state x changes. Such problems can be solved by functional iteration. However, one disadvantage is that it may not be possible to find out the functional form in each iteration.

2.3. The case where volatility depends on the level of the public good

This section will present specific results for the case where $\sigma(u_i, u_j, x)$ is independent of the contribution rates and linearly dependent on the level of public good. The motivation for this specification is that a larger project value generally fluctuates more significantly than a smaller one. More precisely, it can be assumed that:

$$\sigma(u_i, u_j, x) = \sigma x.$$

Equation (2.10) provides the Nash-optimal control $u_i^*(x) = V'(x)$. The Hamilton-Jacobi-Bellman equation is given by

$$\frac{\sigma^2 x^2}{2} V''(x) + \frac{3}{2} [V'(x)]^2 - \delta x V'(x) - rV(x) - bx^2 + ax = 0 \quad (2.12)$$

The finite horizon approximation approach leads to the solution, $Ax^2 + Bx + C$, where A , B and C are fixed points of the following differential equations

$$\begin{aligned} A'(t; T) &= -6A^2(t) + (r + 2\delta - \sigma^2) A(t) + b \\ B'(t; T) &= (r + \delta - 6A(t)) B(t) - a \\ C'(t; T) &= rC(t) - \frac{3}{2} B^2(t) \end{aligned}$$

It can be shown that $A^\pm = \frac{(r+2\delta-\sigma^2) \pm \sqrt{(r+2\delta-\sigma^2)^2 + 24b}}{12}$, $B = \frac{a}{r-6A^\pm + \delta}$ and $C = \frac{3a^2}{2r(r-6A^\pm + \delta)^2}$ are solutions of the fixed point equation for the system above. An analysis of the (A, A') -diagram and the transversality condition for a finite time horizon model with a terminal time T shows that $A(t; T)$ should lie in the interval $[A^-, 0]$. Therefore, as T tends to infinity, $A(t; T)$ converges to $A_p = A^-$. With the notation

$$\begin{aligned} A_p &= \frac{(r + 2\delta - \sigma^2) - \sqrt{(r + 2\delta - \sigma^2)^2 + 24b}}{12} < 0 \\ B_p &= \frac{a}{r - 6A_p + \delta} > 0 \\ C_p &= \frac{3a^2}{2r(r - 6A_p + \delta)^2} > 0 \end{aligned}$$

the value function for this problem is given by:

$$V(x) = A_p x^2 + B_p x + C_p$$

and the Nash-equilibrium strategy is:

$$u_i^*(x) = 2A_px + B_p.$$

Note that the Nash-equilibrium strategy is negative when x is sufficiently large. In this case, agents could stop investing until x has reduced. Under the Nash-equilibrium controls, state x follows the linear stochastic differential equation:

$$dx(t) = (4A_px(t) + 2B_p - \delta x(t)) dt + \sigma x(t) dW(t), \quad x(0) = x_0 \quad (2.13)$$

Note that the process defined by (2.13) always remains positive, since

$$dx(t) = 2B_p dt, \text{ if } x(t) = 0$$

The stochastic differential equation is linear and can be solved analytically. In fact it follows from Kuo (2000) section 11.1 in [?] that equation (2.13) has the solution

$$x(t) = x_0 e^{\left(4A_p - \delta - \frac{\sigma^2}{2}\right)t + \sigma W(t)} + \int_0^t 2B_p e^{\left(4A_p - \delta - \frac{\sigma^2}{2}\right)(t-s) + \sigma(W(t) - W(s))} ds. \quad (2.14)$$

Taking expectations gives

$$E\{x(t)\} = E\left\{x_0 e^{\left(4A_p - \delta - \frac{\sigma^2}{2}\right)t + \sigma W(t)}\right\} + E\left\{\int_0^t 2B_p e^{\left(4A_p - \delta - \frac{\sigma^2}{2}\right)(t-s) + \sigma(W(t) - W(s))} ds\right\}$$

Using the fact that the expectation of a geometric Brownian motion is known and interchanging expectation and integration within the second integral, the following is obtained

$$E\left\{x_0 e^{\left(4A_p - \delta - \frac{\sigma^2}{2}\right)t + \sigma W(t)}\right\} = x_0 e^{(4A_p - \delta)t}$$

and

$$E \left\{ \int_0^t 2B_p e^{\left(4A_p - \delta - \frac{\sigma^2}{2}\right)(t-s) + \sigma(W(t) - W(s))} ds \right\} = \frac{2B_p}{\delta - 4A_p} (1 - e^{(4A_p - \delta)t}).$$

Therefore,

$$\lim_{t \rightarrow \infty} E \{x(t)\} = \frac{2B_p}{\delta - 4A_p} \quad (2.15)$$

Note that, in this case, the expected level of the public good converges to its deterministic analogue. Also note however that A_p and B_p are both dependent on σ and therefore uncertainty has an effect. The slope coefficient in the Nash-equilibrium control $u_i^*(X) = 2A_p X + B_p$ is negative. In the case of a deterministic game model, Fershtman and Nitzan have given an economic interpretation in [?]. The interpretation in the case presented here is basically the same. Once a large value of the project is observed, individuals may try to free ride on other agents, and this leads to a decreasing contribution rate. Now, considering the equilibrium distribution of $x(t)$, i.e.,

$$\lim_{t \rightarrow \infty} x(t).$$

If the density function of the equilibrium distribution of $x(t)$ exists, it will satisfy the Kolmogorov forward equation and is given by the solution for the differential equation

$$\begin{aligned} \frac{\sigma^2 x^2}{2} P''(x) + (2\sigma^2 + \delta - 4A_p) x P'(x) - 2B_p P'(x) + (\sigma^2 + \delta - 4A_p) P(x) &= 0, \\ \int_0^\infty P(x) dx &= 1, \end{aligned}$$

where $P(x)$ is the probability density function. Nevertheless, the Kolmogorov forward equation may not be easily solved due to the condition $\int_0^\infty P(x) dx = 1$. To

solve it, it is supposed that

$$F(y) = \int_0^y P(x)dx$$

Since

$$\begin{aligned} \int_0^y xP'(x)dx &= yF'(y) - F(y) \\ \int_0^y x^2P''(x)dx &= y^2F''(y) - 2yF'(y) + 2F(y) \end{aligned}$$

the Kolmogorov forward equation is equivalent to

$$\begin{aligned} \frac{\sigma^2 y^2}{2} F''(y) + (\sigma^2 + \delta - 4A_p) yF'(y) - 2B_p F'(y) &= 0 \\ F(0) &= 0 \\ \lim_{y \rightarrow \infty} F(y) &= 1 \end{aligned}$$

The Kolmogorov forward equation may not be solved analytically; on the other hand, the transformation leads the Kolmogorov forward equation to be solved numerically via the shooting method, see [?], section 6.1. The concept of shooting method is to estimate the first derivate at 0 and therefore a boundary value problem can be converted to an initial value problem. To solve an initial value problem, the finite difference method or Runge-Kutta algorithm can be applied. Some results of sensitivity analysis will be presented. It can be shown that A_p is increasing in δ . Indeed,

$$\frac{\partial A_p}{\partial \delta} = \frac{-(r + 2\delta - \sigma^2) + \sqrt{(r + 2\delta - \sigma^2)^2 + 24b}}{6\sqrt{(r + 2\delta - \sigma^2)^2 + 24b}} > 0.$$

$r + 2\delta - \sigma^2$ is assumed to be non negative. This condition is satisfied for all realistic values of σ and r . Since $r - 6A_p + \delta$ is increasing, B_p and C_p are decreasing. Therefore,

since the coefficient A_p is dominant for large x while the coefficients B_p and C_p are dominant for small x , it can be seen that an increase in the depreciation rate leads to a lower value of optimal utility for small x and a larger value of optimal utility for large x . The interpretation of this phenomenon is that within this specification in the stochastic case, the example represents a case where it is undesirable to have too large a value of public good. Moreover, since A_p is increasing in δ , the free rider effect is less apparent if the depreciation rate is higher. On the other hand, A_p is decreasing in σ . This follows from

$$\frac{\partial A_p}{\partial \sigma} = \frac{\sigma \left[(r + 2\delta - \sigma^2) - \sqrt{(r + 2\delta - \sigma^2)^2 + 24b} \right]}{6\sqrt{(r + 2\delta - \sigma^2)^2 + 24b}}.$$

Similarly it can be seen that B_p and C_p are decreasing with respect to σ . Thus, the stationary Nash-equilibrium and optimal utility are decreasing in σ too. It can be seen from equation (2.15) that the long term expectation is decreasing with respect to σ , which means that a higher risk reduces the expected value of the project, which in effect causes agents to have even less incentives to invest money. Furthermore, a larger σ leads the free rider effect to be more apparent because the slope of the stationary feedback Nash equilibrium is decreasing in σ . On the other hand, if both agents are allowed to cooperate, i.e., they do not concentrate on their individual utility functions but instead focus on the joint utility function, then the objective functional is given by

$$\max_{u_1, u_2} E \left\{ \int_0^\infty e^{-rt} \left[-\frac{u_1^2(t) + u_2^2(t)}{2} + 2(ax(t) - bx^2(t)) \right] dt \mid x(0) = x_0 \right\}. \quad (2.16)$$

The finite horizon approximation approach leads the Hamilton-Jacobi-Bellman equation to be given by

$$rV(t, x; T) - \frac{\partial}{\partial t}V(t, x; T) = \max_{u_1, u_2} \left\{ -\frac{u_1^2 + u_2^2}{2} + 2(ax - bx^2) + \frac{\partial}{\partial x}V(t, x; T) + \frac{\sigma^2 x^2}{2} \frac{\partial^2}{\partial x^2}V(t, x; T) \right\}. \quad (2.17)$$

A necessary condition of maximizing equation (2.17) is given by

$$u_1^* = u_2^* = \frac{\partial}{\partial x}V(t, x; T). \quad (2.18)$$

Substituting equation (2.18) into equation (2.17), gives

$$rV(t, x; T) - \frac{\partial}{\partial t}V(t, x; T) = \frac{\sigma^2 x^2}{2} \frac{\partial^2}{\partial x^2}V(t, x; T) + \left[\frac{\partial}{\partial x}V(t, x; T) \right]^2 - \delta x \frac{\partial}{\partial x}V(t, x; T) + (ax - bx^2). \quad (2.19)$$

The solution for equation (2.19) is defined by the form $A(t; T)x^2 + B(t; T)x + C(t; T)$. Substituting equation (2.19) into equation (2.18), gives the following ordinary differential equations

$$\begin{aligned} A'(t; T) &= -4A^2(t; T) + (r + 2\delta - \sigma^2) A(t; T) + 2b, \\ B'(t; T) &= (r + \delta - 4A(t; T)) B(t; T) - 2a, \\ C'(t; T) &= rC(t; T) - B^2(t; T). \end{aligned}$$

Next it is necessary to derive the fixed points for the above system. To satisfy equation (2.5), $A(t; T)$, $B(t; T)$ and $C(t; T)$ are convergent to

$$\begin{aligned} A_p^c &= \frac{(r + 2\delta - \sigma^2) - \sqrt{(r + 2\delta - \sigma^2)^2 + 32b}}{8} < 0, \\ B_p^c &= \frac{2a}{r - 4A_p^c + \delta} > 0, \\ C_p^c &= \frac{(B_p^c)^2}{r} > 0. \end{aligned}$$

Now comparing the free rider effect for the cooperative case with the non-cooperative case: as seen above, the free rider effect is affected by the negative slope A_p and A_p^c . A lower slope leads the free rider effect to be more apparent. Since

$$\begin{aligned} A_p^c - A_p &= \frac{(r + 2\delta - \sigma^2) - \sqrt{(r + 2\delta - \sigma^2)^2 + 32b}}{8} - \frac{(r + 2\delta - \sigma^2) - \sqrt{(r + 2\delta - \sigma^2)^2 + 24b}}{12} \\ &= \frac{(r + 2\delta - \sigma^2) - 3\sqrt{(r + 2\delta - \sigma^2)^2 + 32b} + 2\sqrt{(r + 2\delta - \sigma^2)^2 + 24b}}{24} \\ &\leq \frac{(r + 2\delta - \sigma^2) - \sqrt{(r + 2\delta - \sigma^2)^2 + 32b}}{24} \leq 0, \end{aligned}$$

it can be seen that the free rider effect is more apparent under the cooperative condition. On the other hand, it can be proved that $A_p^c \leq \frac{3}{2}A_p$ and this implies that $B_p^c \geq 2B_p$; indeed,

$$\begin{aligned} B_p^c - 2B_p &= \frac{2a}{r - 4A_p^c + \delta} - \frac{2a}{r - 6A_p + \delta} \\ &= \frac{a \left[\sqrt{(r + 2\delta - \sigma^2)^2 + 24b} - \sqrt{(r + 2\delta - \sigma^2)^2 + 32b} \right]}{(r - 4A_p^c + \delta)(r - 6A_p + \delta)} \leq 0. \end{aligned}$$

On the other hand, it can be shown that $A_p^c \geq 2A_p$ and $C_p^c \geq 2C_p$. Therefore, the difference between the joint optimal utility function and the sum of optimal utility functions, $(A_p^c - 2A_p)x^2 + (B_p^c - 2B_p)x + (C_p^c - 2C_p)$, has the minimum $\frac{2B_p - B_p^c}{2(A_p^c - 2A_p)} \leq 0$. This therefore implies that agents prefer to compete if

$$x \in \left[\frac{-(B_p^c - 2B_p) - \sqrt{K}}{2(A_p^c - 2A_p)}, \frac{-(B_p^c - 2B_p) + \sqrt{K}}{2(A_p^c - 2A_p)} \right],$$

when $K \geq 0$ is given by

$$K = (B_p^c - 2B_p)^2 - 4(A_p^c - 2A_p)(C_p^c - 2C_p).$$

It can be seen that agents prefer to cooperate when x is sufficiently large or low. When x is large, agents free ride others more obviously under cooperation and this leads x to decrease significantly. On the other side, a lower x implies a lower risk and therefore agents cooperate under a lower risk. It can be shown that the limit of the expectation of $x(t)$ under cooperation is larger than under non-cooperation. Even though the free rider effect is more apparent when cooperating, the limit of the expectation of the public good value is larger.

A numerical example of this case is presented. This is based on the assumptions that $r = 0.1$, $\delta = 0.3$, $\sigma = 0.2$, $a = 2$ and $b = 1$. To single out the expectation of the corresponding state, it is assumed that $T = 10$ and 20000 trajectories are simulated. In Figure 2.1, it can be seen that if we fix $x = 0$, the optimal utility function is decreasing. On the other hand, if $x = 5$ is fixed, the optimal utility function is increasing. In Figure 2.2, for any fixed x , the optimal utility function is decreasing. In Figure 2.3, it can be seen that the limit is approximately 0.91 and less than the deterministic version, and this is caused by the risk. The difference of the optimal

utility functions are also presented, between the deterministic and stochastic cases, see Figure 2.4. A larger x implies a larger difference due to the volatility σx . On the other hand, to solve the Kolmogorov forward equation numerically, it is assumed that $F(10) = 1$ and this leads to a boundary value problem. The density of the equilibrium distribution of $x(t)$ is presented in Figure 2.5 and it can be seen that the probability density function has the maximum around the limit of 0.91.

2.4. The case where volatility depends on the contribution rate

In this section it is assumed that the risk is independent of the level of the public good, but instead depends on the current contribution rate of the agents, i.e.,

$$\sigma(u_i, u_j) = \sigma \sqrt{u_i + u_j}.$$

The interpretation of this is that an extremely large contribution rate may cause the public good to fluctuate more heavily than under a low contribution rate. This is in many cases a reasonable assumption. According to equation (2.10), a Nash-optimal control then satisfies

$$u_i^* = V'(x) + \frac{\sigma^2}{2} V''(x).$$

Substituting this into the Hamilton-Jacobi-Bellman equation generates the following value function

$$\frac{3\sigma^4}{8} [V''(x)]^2 + \frac{3\sigma^2}{2} V''(x)V'(x) + \frac{3}{2} [V'(x)]^2 - \delta x V'(x) - rV(x) - bx^2 + ax = 0. \quad (2.20)$$

The finite horizon approximation approach leads investor i to obtain the corresponding optimal utility given by $V(x) = A_m x^2 + B_m x + C_m$, where

$$\begin{aligned} A_m &= \frac{(r + 2\delta) - \sqrt{(r + 2\delta)^2 + 24b}}{12} < 0, \\ B_m &= \frac{6\sigma^2 A_m^2 + a}{r - 6A_m + \delta} > 0, \\ C_m &= \frac{3}{2r} (\sigma^4 A_m^2 + 2\sigma^2 A_m B_m + B_m^2). \end{aligned}$$

Note that A_m is independent of σ . The Nash-optimal strategy is then given by

$$u_i^*(x) = 2A_m x + \sigma^2 A_m + B_m.$$

It is assumed that

$$(r + \delta)^2 \sigma^2 + 12a > (r + \delta) \sigma \sqrt{(r + \delta)^2 + 24b},$$

which therefore implies that $\sigma^2 A_m + B_m > 0$. This guarantees that $u_i^*(0) > 0$ and therefore that when the public good level reaches the value 0, agents have a positive contribution rate. In the deterministic case this would be sufficient to guarantee that the level of the public good x remains positive at all times, and therefore that the contribution rate always remains positive and hence admissible. It also guarantees that C_m is positive. In the stochastic case, a large negative random shock produced by the underlying Brownian motion, could formally cause the level x of the public good and hence also the contribution rate $u_i^*(x)$ to become negative. In order to avoid negative contribution rates in this model, it is necessary to assume that $\delta > \frac{4\sigma^2 A_m^2}{\sigma^2 A_m + B_m}$. This is sufficient, as can be seen from the following discussion. The state trajectory corresponding to the strategies $u_i^*(x)$ is the solution of the

following stochastic differential equation

$$dx(t) = [4A_mx(t) + 2B_m + 2\sigma^2 A_m - \delta x(t)] dt + \sigma \sqrt{4A_mx(t) + 2B_m + 2\sigma^2 A_m} dW(t) \quad (2.21)$$

Under the affine transformation $z(t) = 4A_mx(t) + 2B_m + 2\sigma^2 A_m = 2u_i^*(t)$ the solution of process (2.21) becomes a Cox-Ingersoll-Ross process, i.e.

$$\begin{aligned} dz(t) &= \kappa(\theta - z(t))dt + \nu\sqrt{z(t)}dW(t), \\ z(0) &= 4A_mx_0 + 2\sigma^2 A_m + 2B_m. \end{aligned}$$

with $\kappa = (\delta - 4A_m)$, $\theta = \frac{2\delta\sigma^2 A_m + 2\delta B_m}{\delta - 4A_m}$ and $\nu = 4\sigma A$. It is well known, Alos and Ewald (2008) [?], that positivity is guaranteed by the condition $2\kappa\theta > \nu^2$ which leads exactly to the condition on δ . As $z(t) = 2u_i^*(t)$ this then guarantees that along any realized path the contribution rate toward the public good is always positive. Any Cox-Ingersoll-Ross process is mean reverting to θ with mean reversion speed κ . This process is mean reverting to $\frac{2\delta\sigma^2 A_m + 2\delta B_m}{\delta - 4A_m}$ with mean reversion speed $\delta - 4A_m$. From this it can be seen that the parameters A_m and δ play an important role in how fast the public good level will converge to its mean reversion level, leaving uncertainty effects aside. Equation (2.6) can be rewritten as

$$dw(t) = (\delta - 4A_c) \left(\frac{2\delta B_c}{\delta - 4A_c} - w(t) \right) dt$$

where $w(t) = 4A_c x(t) + 2B_c$. It can be seen that the mean reversion speed of the deterministic version is also given by $\delta - 4A_m$ since $A_c = A_m$. On the other hand, it can be concluded from [?], page 309, that the density function of the equilibrium

distribution of $z(t)$ is given by

$$P_Z(y) = \left(\frac{2\beta}{\gamma^2}\right)^{\frac{2\alpha}{\gamma^2}} \frac{1}{\Gamma\left(\frac{2\alpha}{\gamma^2}\right)} y^{\frac{2\alpha-\gamma^2}{\gamma^2}} e^{-\frac{2\beta}{\gamma^2}y}$$

where $\alpha = 2\delta(\sigma^2 A_m + B_m)$, $\beta = \delta - 4A_m$, $\gamma = 4\sigma A_m$ and $\Gamma(x)$ is the gamma function. The density function for the equilibrium distribution of $x(t)$ is given by

$$P_X(x) = -4A_m P_Z(4A_m x + 2B_m + 2\sigma^2 A_m)$$

Note that $z(t) = 2u_i^*(t)$ and therefore the density function of the equilibrium distribution of $u_i^*(t)$ can be easily derived. Moreover, $u_i^*(t)$ is always non-negative and the value of the good is bounded above. In the long term, the mean reversion level determines the expectation, thus obtaining

$$\lim_{t \rightarrow \infty} E\{x(t)\} = \frac{2B_m + 2\sigma^2 A_m}{\delta - 4A_m}. \quad (2.22)$$

The process (2.21) describing the public good level under the equilibrium strategies in the framework of this section is much more regular than the process described in (2.13). In fact process (2.21) admits a stationary distribution for which analytical formulas exist. In contrast to (2.13) the variance of (2.21) is bounded. This can be interpreted as reduced uncertainty, once the equilibrium strategies have been adopted. It can be seen that the expectation is functionally different from the type of volatility given by σx and the case of the deterministic model in section 2.1. Note however, that the Nash-equilibrium is also linear with a negative slope coefficient. Next, the impact of the depreciation rate will be studied. For this, it is assumed that each equation is a function of δ . It can be shown that A_m is increasing and

B_m is decreasing in δ . Indeed,

$$\frac{\partial}{\partial \delta} A_m = \frac{1}{6} \left[1 - \frac{r + 2\delta}{\sqrt{(r + 2\delta)^2 + 24b}} \right] > 0$$

and

$$\frac{\partial}{\partial \delta} B_m = \frac{12\sigma^2 A_m \frac{\partial}{\partial \delta} A_m - 2 \left(1 - 6 \frac{\partial}{\partial \delta} A_m \right) (6\sigma^2 A_m^2 + a)}{r - 6A_m + \delta} < 0.$$

Similarly to the case of σx , a larger depreciation rate leads the free rider effect to be less apparent. On the other hand, because A_m is independent of σ , the free rider effect is not affected by the risk, which means that in this case it can be analogous to the deterministic version. Note that a positive long-term expected value for the public good level can be guaranteed by the condition $B_m > -\sigma^2 A_m$. Economically, it is easy to see that each investor does not have any incentive to invest into this project, if the expectation of it is negative. This assumption also implies a positive C_m . The Nash-equilibrium is a linear function with a negative slope and therefore reinforces the free rider effect. By computing the first derivative of equation (2.22) one can see that it is decreasing with δ . It can also be seen that a larger value of depreciation rate leads to a lower project value. On the other hand, it can be shown that the expectation of the corresponding state is less than in the case of the deterministic model considered in section 2.1. The difference is in fact given by

$$\frac{2\sigma^2 A_m (r + \delta)}{(\delta - 4A_m)(r - 6A_m + \delta)} < 0.$$

A larger value of σ therefore implies a lower value for the expectation of the project value. Furthermore B_m and C_m as functions of σ are increasing. In the utility function, however, they are still offset by the effect on A_m and a higher value of σ implies a lower contribution rate under the Nash-optimal strategy, which means

both agents invest less if the uncertainty is rising. This can be proved by computing the derivative of u_i^* with respect to σ , which is given by

$$\frac{d}{d\sigma}u_i^*(x) = 2\sigma A_m + \frac{d}{d\sigma}B_m = \frac{2\sigma(r + \delta)A_m}{r - 6A_m + \delta} < 0$$

The interpretation of this is that both agents do have less investment incentive due to higher uncertainty. On the other hand, it also leads to a lower project value. Now, moving to the case where agents cooperate, the joint objective functional is given by equation (2.16) and the finite horizon approximation leads the Hamilton-Jacobi-Bellman equation to be defined by

$$\begin{aligned} & rV(t, x; T) - \frac{\partial}{\partial t}V(t, x; T) \\ &= \max_{u_1, u_2} \left\{ -\frac{u_1^2 + u_2^2}{2} + 2(ax - bx^2) + \frac{\partial}{\partial x}V(t, x; T)(u_1 + u_2 - \delta x) + \frac{\sigma^2(u_1 + u_2)}{2} \frac{\partial^2}{\partial x^2}V(t, x; T) \right\}. \end{aligned} \quad (2.23)$$

A necessary condition for maximizing equation (2.23) is given by

$$u_1^* = u_2^* = \frac{\partial}{\partial x}V(t, x; T) + \frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2}V(t, x; T). \quad (2.24)$$

Substituting equation (2.24) into (2.23), gives

$$\begin{aligned} & rV(t, x; T) - \frac{\partial}{\partial t}V(t, x; T) \\ &= \frac{\sigma^4}{4} \left[\frac{\partial^2}{\partial x^2}V(t, x; T) \right]^2 + \sigma^2 \frac{\partial}{\partial x}V(t, x; T) \frac{\partial^2}{\partial x^2}V(t, x; T) + \left[\frac{\partial}{\partial x}V(t, x; T) \right]^2 - \delta x \frac{\partial}{\partial x}V(t, x; T) \\ & \quad + 2ax - 2bx^2. \end{aligned} \quad (2.25)$$

The solution for equation (2.25) is given by $V(t, x; T) = A(t; T)x^2 + B(t; T)x + C(t; T)$. Substituting the solution into equation (2.25), gives

$$\begin{aligned} A'(t; T) &= -4A(t; T)^2 + (r + 2\delta) A(t; T) + 2b, \\ B'(t; T) &= [r - 4A(t; T) + \delta] B(t; T) - 4\sigma^2 A^2(t; T) - 2a, \\ C'(t; T) &= rC(t; T) - \sigma^4 A^2(t; T) - 2\sigma^2 A(t; T)B(t; T) - B^2(t; T). \end{aligned}$$

The finite horizon approximation leads $A(t; T)$, $B(t; T)$ and $C(t; T)$ to converge to $A_m^c = \frac{(r+2\delta) - \sqrt{(r+2\delta)^2 + 32b}}{8}$, $B_m^c = \frac{4\sigma^2(A_m^c)^2 + 2a}{r - 4A_m^c + \delta}$ and $C_m^c = \frac{1}{r} [\sigma^4 (A_m^c)^2 + 2\sigma^2 A_m^c B_m^c + (B_m^c)^2]$, respectively. Note that A_m^c is not dependent on σ , which means that no matter how large σ is, the free rider effect is not affected. Since $A_p = A_m$ and $A_p^c = A_m^c$ when $\sigma = 0$, $A_m^c \leq A_m$, which means that the free rider effect is more apparent under cooperation. On the other hand, when x is sufficiently large, the joint optimal utility function is larger than the sum of optimal utility functions. However, due to the property of the public good, agents do not prefer a larger x . Because they free ride others heavily when x is large, it is realistic that agents cooperate to reduce x . It can be seen that the limit of expectation for the corresponding state under cooperation is larger than under non-cooperation. Indeed,

$$\frac{2B_m^c + 2\sigma^2 A_m^c}{\delta - 4A_m^c} - \frac{2B_m + 2\sigma^2 A_m}{\delta - 4A_m} = \frac{2[\delta(B_m - B_m^c) + \delta\sigma^2(A_m - A_m^c) + 4(A_m B_m^c - A_m^c B_m)]}{(\delta - 4A_m)(\delta - 4A_m^c)}$$

is increasing in σ and the case that $\sigma = 0$ has been proved in the previous section.

Numerical results are provided for this case using the same parameters as in section 2.3. In Figure 2.6 similar behaviour can be observed to the case of level dependent volatility, i.e., σx . In Figure 2.7, it can be seen that for any fixed x , the optimal utility function is increasing. In Figure 2.8, the expectation converges to around 0.92; in contrast with the case of σx , it is closer to the maximum of $\frac{a}{2b}$.

The difference of the optimal utility functions between deterministic and stochastic versions is represented in Figure 2.9. It can be seen that the difference is less than in the case of level dependent volatility. This is realistic because agents can reduce the uncertainty by choosing their contribution rate. On the other side, comparison with Figure 2.4 shows that a lower x implies a larger difference while in Figure 2.9, this is reversed. The density function of the equilibrium distribution $x(t)$ is presented in Figure 2.10, and similar to Figure 2.5, the probability density function has the maximum at 0.92.

2.5. Conclusions

Fershtman and Nitzan's model [?] has been successfully extended by introducing two different uncertainty effects. The first effect is due to level dependent volatility, while the second effect is due to contribution related volatility. In both cases analytical solution of the Nash-equilibrium strategies understood as stationary Markovian Nash-equilibria strategies have been computed. This analysis also shows that uncertainty affects the strategies of the agents. In opposition to the idea that an increase in uncertainty may reduce the free rider effect, it was found that in fact in the case where volatility is dependent on the level of the public good, the free rider effect is emphasized by uncertainty. Lastly, it was evident that under the same level of public good in both cases, the free rider effect is more apparent when agents are allowed to cooperate. These results might be of particular interest in public insurance.

Graphical Illustration

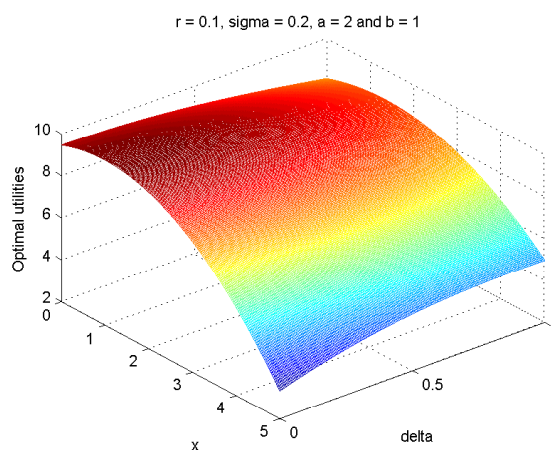


FIGURE 2.1: Value function in terms of x and δ for volatility depending on the level of the public good under $r = 0.1$, $\sigma = 0.2$, $a = 2$ and $b = 1$

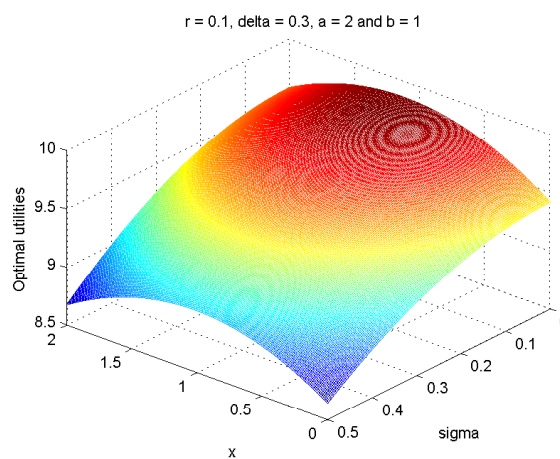


FIGURE 2.2: Value function in terms of x and σ for volatility depending on the level of the public good under $r = 0.1$, $\delta = 0.3$, $a = 2$ and $b = 1$

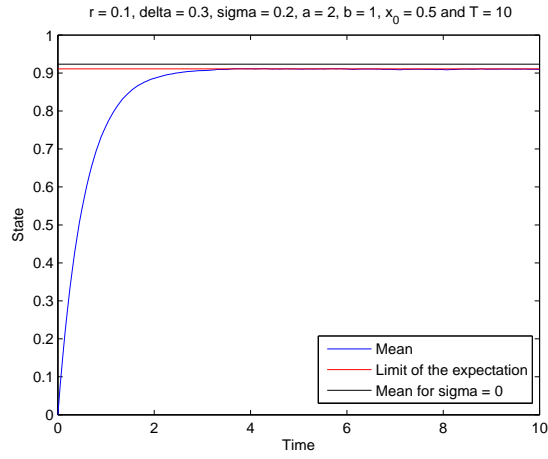


FIGURE 2.3: State and limit for volatility depending on the level of the public good under $r = 0.1$, $\delta = 0.3$, $\sigma = 0.2$, $a = 2$, $b = 1$ and $T = 10$

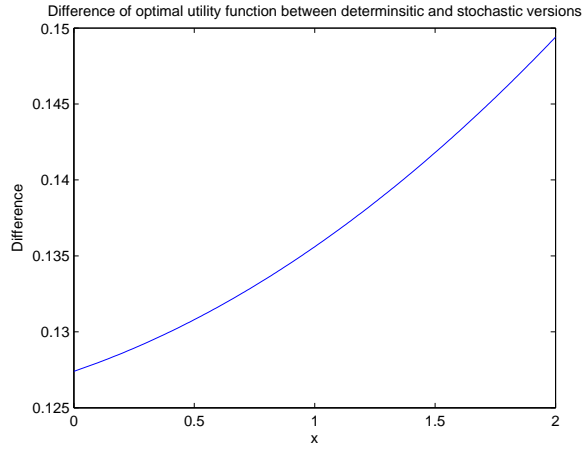


FIGURE 2.4: Difference of value functions between deterministic and stochastic version for volatility depending on the level of the public good under $r = 0.1$, $\delta = 0.3$, $\sigma = 0.2$, $a = 2$ and $b = 1$

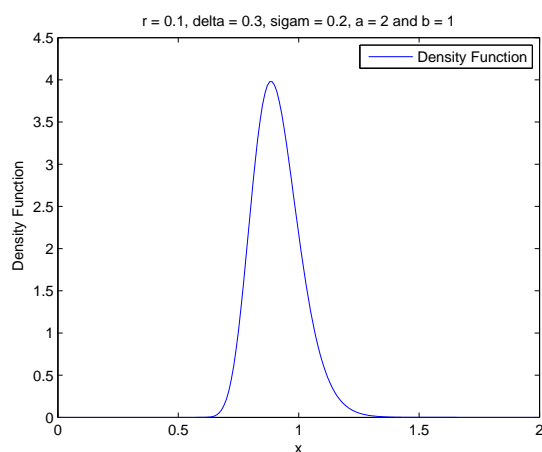


FIGURE 2.5: The Density function of the equilibrium distribution of $x(t)$ for the volatility depending on the level of the public good under $r = 0.1$, $\delta = 0.3$, $\sigma = 0.2$, $a = 2$ and $b = 1$

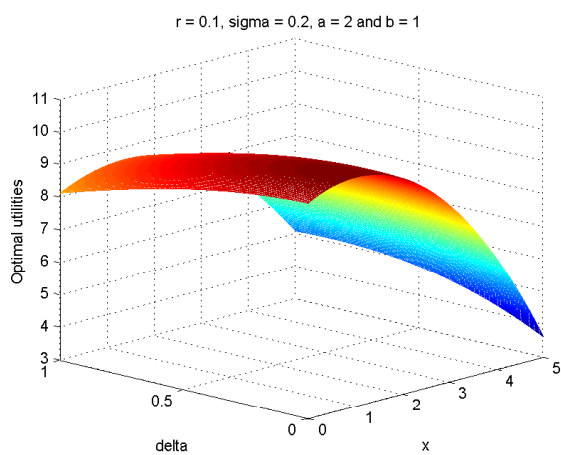


FIGURE 2.6: Value function in terms of x and δ for volatility depending on the contribution rate under $r = 0.1$, $\sigma = 0.2$, $a = 2$ and $b = 1$

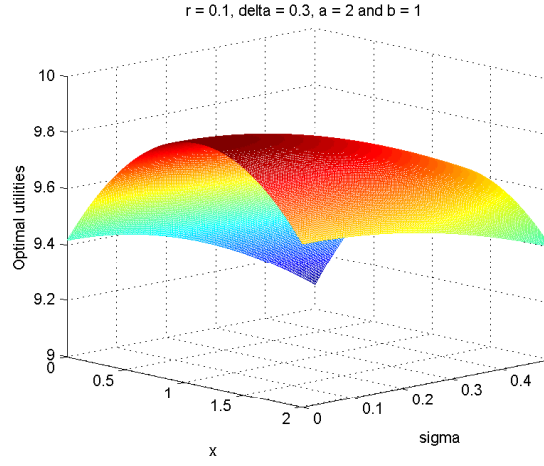


FIGURE 2.7: Value function in terms of x and σ for volatility depending on the contribution rate under $r = 0.1$, $\delta = 0.3$, $a = 2$ and $b = 1$

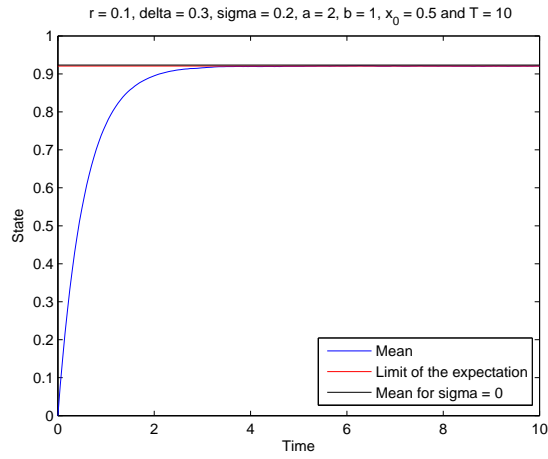


FIGURE 2.8: State and limit for volatility depending on the contribution rate under $r = 0.1$, $\delta = 0.3$, $\sigma = 0.2$, $a = 2$, $b = 1$ and $T = 10$

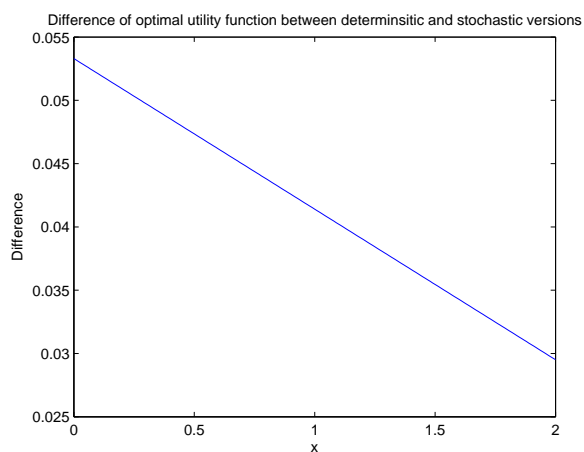


FIGURE 2.9: Difference of value functions between deterministic and stochastic version for volatility depending on the contribution rate under $r = 0.1$, $\delta = 0.3$, $\sigma = 0.2$, $a = 2$ and $b = 1$

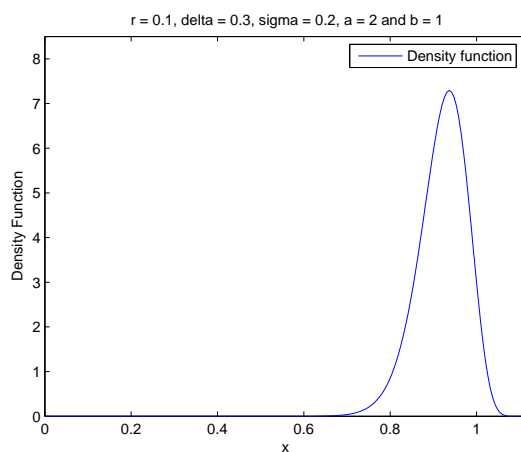


FIGURE 2.10: The density function of the equilibrium distribution of $x(t)$ for volatility depending on the contribution rate under $r = 0.1$, $\delta = 0.3$, $\sigma = 0.2$, $a = 2$ and $b = 1$

Chapter 3

A Stochastic Differential Fishery Game for a Two Species Fish Population with Ecological Interaction

The first mathematical models of fisheries were developed in the 1950's with the works of Gordon (1954) [?] and Schaefer (1957) [?]. Clark (1976) built on these early models and considered the conflict between agents fishing for the same species in a dynamic game model, see [?]. Various recent works focus on continuous time and are based on differential game models. Among them are Dockner et al. (1989) [?], Haurie et al. (1994) [?], Jorgensen and Yeung (1996) [?] and Kaitala (1989) [?]. All consider the case of a single species. Kaitala (1989) presented a deterministic game model in which he assumed that the price of the species is given by a constant p while costs for each agent i are defined by a proportionality constant c_i , see [?]. He studied the competitive case and derived a feedback Nash-equilibrium. Hamalainen,

Haurie and Kaitala (1985) studied a slightly different model and derived an open-loop Nash equilibrium, see [?]. The authors also considered the cooperative case and provided two numerical examples. Dockner et al. (1989) considered a continuous time framework involving uncertainty, see [?]. They studied the non cooperative case and followed the concept used in Clark (1980) [?] and Simaan et al. (1978) [?] for the specification of a price function. Dockner et al. assumed that the price depends on the quantity harvested by all agents. Furthermore, Dockner et al. derived not only a feedback Nash-equilibrium but also a Stackelberg-equilibrium. Hamalainen et al. (1994) [?] studied the cooperative case and Haurie (1991) (1993) extended on these ideas in [?] and [?] to model the triggering mechanism as a Markov jump process and the retaliation duration as an exponential random variable. Jorgensen and Yeung (1996) studied a model where the concept of price is similar to [?] and costs are not constant, but depend on a function which is decreasing in the stock of biomass, see [?]. These authors derived a feedback Nash equilibrium and also considered the cooperative case. Moreover, they also analyzed surplus maximization and optimal market size.

This chapter will examine a model where fishery agents compete against each other to fish for two different species. Ecologically, these two species are assumed to interact with each other. The assumed interactions include the cases of predator, prey and competition. The model is assumed to be time homogeneous and each agent to be facing an infinite time horizon. The case of a two species predatory game theoretic fishery model has been studied before by Quirk and Smith (1977) [?] and Anderson (1975) [?] and furthermore by Sumaila (1997) [?]. Sumaila focused on the case of the Barents Sea and the species Cod and Capelin. In this particular case, Cod preys on Capelin and Sumaila provides various results which document the importance of studying fisheries within a general multi-species ecological context.

Without this realization, game theoretic models will produce inefficient fishery strategies. In particular Sumaila compares the situation where two fisheries are managed by their individual owners with fishing rights exclusive for one particular species, with the case of joint management or competitive fishing of both species. He showed that the ecological interaction of the two species has a significant economical effect on the fisheries.

The models considered by Sumaila (1998), Quirk and Smith (1977) and Anderson (1975) are neither continuous time, nor do they include ecological uncertainty. Both of these aspects are, however, crucial for the setup of a realistic dynamic model. The aspect of continuous time can, in principle, be mimicked by using a discrete time model with small enough time steps, but the case presented here allows computation of semi analytic solutions which are numerically tractable. The inclusion of ecological uncertainty is fundamental and new in this context. In a world of climate change and increasing sea temperatures with unpredictable effects, the author and his supervisor consider it to be an absolutely necessary. A consequence of including ecological interaction in this way is, of course, that mathematically the model becomes far more challenging than corresponding single species models. The analysis requires solution of partial differential equations including two state variables, rather than a differential equation depending on single state variable in the time homogeneous case. Nevertheless it has been possible to solve the model semi-analytically, by which it is meant that the author and his supervisor have derived explicit forms for the strategies and value function which depend on certain constants, which can be computed numerically by an iterative process. Following this, there is a discussion of how different parameters affect the solution and their economical interpretation. In addition to the setup where each fishery can harvest both species, the case where fisheries are restricted to harvest

only one of the two species is examined, as well as the case where the fisheries cooperate, e.g. are jointly managed. In both cases, the economic consequences are highlighted, along with the ecological impact. One significant observation is that a competitive ecological system may thrive better under cooperative management, while a predatory ecological system may thrive better under competition between the fisheries. These results are considered to have very striking policy implications.

The remainder of the chapter is organized as follows: in section 3.1, there is a brief review of a typical deterministic model of two ecologically interacting species and, based on this set up, the current game theoretic model which includes uncertainty. This model will be analyzed to derive a feedback Nash equilibrium by the Hamilton-Jacobi-Bellman approach. Section 3.2 contains a detailed sensitivity analysis. In section 3.3, the case where regulation restricts each fishery to concentrate on one particular species is examined, together with the economic inefficiencies and consequences resulting from this. Section 3.4 introduces the concept of cooperation between the fisheries. Section 3.5 presents the numerical results and the main conclusions are summarized in section 3.6.

3.1. A stochastic differential fishery game with ecological interaction

The case of deterministic combined predator-prey and competitive interactions will first be briefly reviewed. Hofbauer and Sigmund (1998) have presented various examples of such dynamic systems, see in particular [?], page 11 for a predator-prey dynamic and page 26 for a competitive dynamic. These two models have been combined and modified accordingly with the aim of achieving an analytically

tractable model. The result is the following dynamics:

$$\begin{aligned} x_1'(t) &= x_1(t) \left[\frac{\alpha_1}{\sqrt{x_1(t)}} - \beta_1 - \gamma_1 \sqrt{\frac{x_2(t)}{x_1(t)}} \right] \\ x_2'(t) &= x_2(t) \left[\frac{\alpha_2}{\sqrt{x_2(t)}} - \beta_2 - \gamma_2 \sqrt{\frac{x_1(t)}{x_2(t)}} \right]. \end{aligned} \quad (3.1)$$

In this interpretation $x_i(t)$ represents the biomass of species i at t . All coefficients are assumed to be constant. The birth rate of species j is given by $\frac{\alpha_j}{\sqrt{x_j}}$ and the death rate by β_j . There are effects on the size of biomass of both species due to predation of one species on the other, as well as competition. The function

$$F_j(x_j) = x_j \left[\frac{\alpha_j}{\sqrt{x_j}} - \beta_j - \gamma_j \sqrt{\frac{x_j'}{x_j}} \right]$$

is referred to as the natural growth function for species $j = 1, 2$. The case of a pure predator-prey system where species 2 hunts species 1 is represented $\alpha_1 > 0$, $\alpha_2 < 0$, $\beta_1 = \beta_2 = 0$, $\gamma_1 < 0$ and $\gamma_2 > 0$, compare [?] equation (2.1). A purely competitive system is represented by the case where all constant are positive. Four stationary points can be derived from this dynamic and are given by $(0, 0)$, $\left(\frac{\alpha_1^2}{\beta_1^2}, 0\right)$, $\left(0, \frac{\alpha_2^2}{\beta_2^2}\right)$ and $\left(\left(\frac{\alpha_1\beta_2 - \alpha_2\gamma_1}{\beta_1\beta_2 - \gamma_1\gamma_2}\right)^2, \left(\frac{\alpha_1\gamma_2 - \alpha_2\beta_1}{\beta_1\beta_2 - \gamma_1\gamma_2}\right)^2\right)$. Linearization of system (4.1) around the latter gives

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} \frac{\alpha_1}{2\sqrt{x_1^*}} - \beta_1 - \frac{\gamma_1\sqrt{x_2^*}}{2\sqrt{x_1^*}} & -\frac{\gamma_1\sqrt{x_1^*}}{2\sqrt{x_2^*}} \\ -\frac{\gamma_2\sqrt{x_2^*}}{2\sqrt{x_1^*}} & \frac{\alpha_2}{2\sqrt{x_2^*}} - \beta_2 - \frac{\gamma_2\sqrt{x_1^*}}{2\sqrt{x_2^*}} \end{pmatrix} \begin{pmatrix} x_1 - x_1^* \\ x_2 - x_2^* \end{pmatrix} \quad (3.2)$$

where $(x_1^*, x_2^*) = \left(\left(\frac{\alpha_1\beta_2 - \alpha_2\gamma_1}{\beta_1\beta_2 - \gamma_1\gamma_2}\right)^2, \left(\frac{\alpha_1\gamma_2 - \alpha_2\beta_1}{\beta_1\beta_2 - \gamma_1\gamma_2}\right)^2\right)$. The other fixed points will not be analysed because, in these cases, at least one species has become extinct, and in

this case the resulting model is identical to a single species model. The eigenvalues of the matrix in system (3.2) are computed via

$$e^{\pm} = \frac{(a_{11} + a_{22}) \pm \sqrt{(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - \gamma_1\gamma_2)}}{2}$$

where

$$\begin{aligned} a_{11} &= \frac{\alpha_1}{2\sqrt{x_1^*}} - \beta_1 - \frac{\gamma_1\sqrt{x_2^*}}{2\sqrt{x_1^*}} \\ a_{22} &= \frac{\alpha_2}{2\sqrt{x_2^*}} - \beta_2 - \frac{\gamma_2\sqrt{x_1^*}}{2\sqrt{x_2^*}}. \end{aligned}$$

Note that $\sqrt{(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - \gamma_1\gamma_2)}$ is always positive. If e^{\pm} are both negative, then the fixed point is asymptotically stable and both species survive. This is the case of stable coexistence; see Figure 3.1. Conversely, if $e^+ > 0$ and $e^- < 0$, the fixed point is unstable. Without intervention, one of the two species will eventually become extinct. This case is called a bistable case, see Figure 3.2.

When introducing the effect of fisheries harvesting, both populations introduce ecological uncertainty into the system (3.1). Jorgensen et al. (1989) (1996), see [?] and [?], regarded the control u_i for agent i as the harvest rate of species i . Alternatively, Kaitala et al. (1994) (1989) defined the control for i as the fishing effort, see also [?] and [?]. This analysis will follow the line of Jorgensen et al. and define the control as the harvest rate for each agent. The state equations for the

biomass of the two species are then given by

$$\begin{aligned} dx_1(t) &= \left\{ x_1(t) \left[\frac{\alpha_1}{\sqrt{x_1(t)}} - \beta_1 - \gamma_1 \sqrt{\frac{x_2(t)}{x_1(t)}} - \sum_{i=1}^N u_i^1(t) \right] dt + \sigma_1 x_1(t) dW_1(t) \right. \\ dx_2(t) &= \left\{ x_2(t) \left[\frac{\alpha_2}{\sqrt{x_2(t)}} - \beta_2 - \gamma_2 \sqrt{\frac{x_1(t)}{x_2(t)}} - \sum_{i=1}^N u_i^2(t) \right] dt + \sigma_2 x_2(t) dW_2(t) \right\} \end{aligned} \quad (3.3)$$

where α_i , β_i , γ_i and σ_i have the same interpretation as before and N is the number of fishery agents. $W_1(t)$ and $W_2(t)$ are two Wiener processes. For simplicity it is assumed that $W_1(t)$ and $W_2(t)$ are uncorrelated. The harvest rate of species j adopted by agent i is denoted as u_i^j . Note that $x_j = 0$ forces $u_i^j = 0$ because it is impossible to harvest an extinct species. This poses natural restrictions on the set of admissible controls. Each agent tries to maximize his or her objective functional, given by

$$\max_{u_i^1, u_i^2} E \left\{ \int_0^\infty e^{-rt} \sum_{j=1}^2 [P^j(t) u_i^j(t) - C_i^j(t) u_i^j(t)] dt \mid x_1(0) = x_{10}, x_2(0) = x_{20} \right\} \quad (3.4)$$

where r is a discount rate. $P^j(t)$ and $C_i^j(t)$ are price and cost functions of species j for agent i , respectively. It is proposed that the price function is given by

$$P^j(t) = \frac{1}{\sqrt{p_j \sum_{i=1}^N u_i^j(t)}}, \quad j = 1, 2 \quad (3.5)$$

and the cost function for agent i is defined as

$$C_i^j(t) = \frac{c_i^j}{\sqrt{x_j(t)}}, \quad j = 1, 2 \quad (3.6)$$

where p_j and c_i^j are constants. that this uses the same cost functions as Jorgensen and Yeung (1996) [?] and the price functions have been multiplied by $\frac{1}{\sqrt{p_j}}$ because this is a two species model. Equation (3.5) implies that prices decrease as the amount of total harvest increases. This accounts for the relationship between supply and demand. On the other hand, in equation (3.6), a larger value of stock of biomass j implies a lower value of costs for i . This assumption is also realistic. It is proposed that $c_i^j = c_j$ and this leads the model to be time homogeneous and symmetric. Under this perspective the objective will be to identify a stationary symmetric feedback Nash-equilibrium. Within a time homogeneous framework, the value function satisfying the Hamilton-Jacobi-Bellman equation does not explicitly depend on time. While this reduces the dimension of the PDE by one, it brings with it the loss of the terminal condition, which helps to identify the value function. Classically, a so called transversality condition is employed, see [?], page 124. In most cases this transversality condition can, however, only be effectively used when the structure of the value function is known. In a framework which relies on numerical solution of the Hamilton-Jacobi-Bellman equation, the transversality condition is in many cases not practicable. Therefore, the finite horizon approximation approach is employed again, which was introduced in section 1.2 and applied in the previous chapter. Now supposing that T is a finite terminal time. Then the Hamilton-Jacobi-Bellman equation for agent i is given by

$$rV_i(t, x; T) - \frac{\partial}{\partial t}V_i(t, x; T) = \max_{u_i^1, u_i^2} \left\{ \sum_{j=1}^2 \left[\frac{u_i^j}{\sqrt{p_j \left(u_i^j + \sum_{k \neq i} u_k^{j*} \right)}} - \frac{c_j u_i^j}{\sqrt{x_j}} + \frac{\sigma_j^2 x_j^2}{2} \frac{\partial^2}{\partial x_j^2} V_i(t, x; T) \right. \right. \\ \left. \left. + \frac{\partial}{\partial x_j} V_i(t, x; T) \left(\alpha_j \sqrt{x_j} - \beta_j x_j - \gamma_j \sqrt{x_1 x_2} - u_i^j - \sum_{k \neq i} u_k^{j*} \right) \right] \right\} \quad (3.7)$$

where $x = (x_1, x_2)$. A necessary condition for maximizers u_i^{j*} of the right hand side of (4.7) is

$$u_i^{j*} = \frac{(2N-1)^2}{4p_j N^3 \left(\frac{c_j}{\sqrt{x_j}} + \frac{\partial}{\partial x_j} V_i(t, x; T) \right)^2}, \quad j = 1, 2. \quad (3.8)$$

Substituting equation (4.8) into equation (4.7), produces the following partial differential equation

$$\begin{aligned} rV_i(t, x; T) = \sum_{j=1}^2 & \left[\frac{2N-1}{2p_j N^2 \left(\frac{c_j}{\sqrt{x_j}} + \frac{\partial}{\partial x_j} V_i(t, x; T) \right)} - \frac{c_j (2N-1)^2}{4p_j N^3 \sqrt{x_j} \left(\frac{c_j}{\sqrt{x_j}} + \frac{\partial}{\partial x_j} V_i(t, x; T) \right)^2} \right. \\ & + \frac{\partial}{\partial x_j} V_i(t, x; T) \left(\alpha_j \sqrt{x_j} - \beta_j x_j - \gamma_j \sqrt{x_1 x_2} - \frac{(2N-1)^2}{4p_j N^2 \left(\frac{c_j}{\sqrt{x_j}} + \frac{\partial}{\partial x_j} V_i(t, x; T) \right)^2} \right) \\ & \left. + \frac{\sigma_j^2 x_j^2}{2} \frac{\partial^2}{\partial x_j^2} V_i(t, x; T) \right] + \frac{\partial}{\partial t} V_i(t, x; T) \end{aligned} \quad (3.9)$$

with boundary condition given by

$$V_i(T, x; T) = 0 \quad (3.10)$$

Following the solution form provided by Jorgensen and Yeung [?], it is assumed that the solution of (3.9) with boundary condition (3.10) is of the following type

$$V_i(t, x; T) = A_1(t)\sqrt{x_1} + A_2(t)\sqrt{x_2} + A_3(t). \quad (3.11)$$

Note that if $W_1(t)$ and $W_2(t)$ are correlated, the solution does not follow the form (3.11). Substituting equation (3.11) into equation (3.9), results in a system of

ordinary differential equations

$$\begin{aligned}
A_1'(t; T) &= k_1 A_1(t; T) - \frac{2N-1}{p_1 N^2 (2c_1 + A_1(t; T))} + \frac{(2c_1 + N A_1(t; T)) (2N-1)^2}{2 p_1 N^3 (2c_1 + A_1(t; T))^2} + \frac{\gamma_2 A_2(t; T)}{2}, \\
A_2'(t; T) &= k_2 A_2(t; T) - \frac{2N-1}{p_2 N^2 (2c_2 + A_2(t; T))} + \frac{(2c_2 + N A_2(t; T)) (2N-1)^2}{2 p_2 N^3 (2c_2 + A_2(t; T))^2} + \frac{\gamma_1 A_1(t; T)}{2}, \\
A_3'(t; T) &= r A_3(t; T) - \frac{\alpha_1 A_1(t; T) + \alpha_2 A_2(t; T)}{2},
\end{aligned} \tag{3.12}$$

where $k_j = \left(r + \frac{\beta_j}{2} + \frac{\sigma_j^2}{8}\right)$. First considering the case of a two species competitive ecological system, i.e., all coefficients are positive. To apply the finite horizon approximation approach, it is necessary to find out the fixed points of the system (3.12) and examine which one satisfies equation (3.10). To derive the fixed points, the following polynomial system must be solved

$$k_1 A_1 - \frac{2N-1}{p_1 N^2 (2c_1 + A_1)} + \frac{(2c_1 + N A_1) (2N-1)^2}{2 p_1 N^3 (2c_1 + A_1)^2} + \frac{\gamma_2 A_2}{2} = 0 \tag{3.13}$$

$$k_2 A_2 - \frac{2N-1}{p_2 N^2 (2c_2 + A_2)} + \frac{(2c_2 + N A_2) (2N-1)^2}{2 p_2 N^3 (2c_2 + A_2)^2} + \frac{\gamma_1 A_1}{2} = 0 \tag{3.14}$$

$$r A_3 - \frac{\alpha_1 A_1 + \alpha_2 A_2}{2} = 0 \tag{3.15}$$

In equation (3.15), A_3 can be easily derived from A_1 and A_2 . So the solution starts by concentrating on equation (3.13) and (3.14). Note that if A_2 is fixed in $[0, X_2)$, with

$$X_2 = \frac{2N-1}{2c_1 \gamma_2 p_1 N^3},$$

equation (3.13) is a cubic polynomial and it can be shown that it has a unique positive solution. Existence can be proved by the intermediate value theorem and uniqueness can be shown by contradiction. A similar argument works for equation

(3.14) if an A_1 is given and lies in $[0, X_1)$, where X_1 is defined by

$$X_1 = \frac{2N - 1}{2c_2\gamma_1p_2N^3}.$$

In [?], the authors claimed that the solution is given by the form $A\sqrt{x} + B$ and A is positive to guarantee that the value function is concave. However, this model, has a more complex dynamic with interactions between the two species and it is not clear why the value function should be concave, neither mathematically nor economically. On the other hand, it is assumed that fishery agents can benefit more from species j if the biomass is larger and this guarantees that A_j is positive. The following proposition says that the system of equations (3.13) and (3.14) have a unique pair of positive solutions.

Proposition 3.1.1. *(Competitive Case) Equations (3.13) and (3.14) have a unique pair of positive solutions (A_1, A_2) , if, for $(j, j') = (1, 2)$ and $(2, 1)$, one of the following conditions holds:*

$$\begin{aligned} \text{Condition (1).} \quad & 4c_1c_2p_{j'}\gamma_j \leq \frac{(2N - 1)(2N - 3)}{N^2} \\ \text{Condition (2).} \quad & X_j \geq \frac{-2c_jk_j + \sqrt{\frac{2c_1c_2p_{j'}\gamma_j}{p_j} - \frac{(2N-1)(2N-3)}{2p_jN^2}}}{k_j} \end{aligned}$$

with X_j defined above.

Proof. For the existence, a constructive proof is presented, as within the numerical analysis it is necessary to compute a pair of solutions, and the method described in this proof is used. Starting with $X'_j = \frac{2N-1}{2c_j\gamma_{j'}p_jN^3}$, where $(j, j') = (1, 2)$ and $(2, 1)$.

Consider the polynomials

$$\begin{aligned} f_j(x) &= k_j x^3 + 4c_j k_j x^2 + \left[4c_j^2 k_j + \frac{(2N-1)(2N-3)}{2p_j N^2} \right] x + \frac{c_j(1-2N)}{p_j N^3} \\ g_j(x) &= \frac{\gamma_{j'}}{2} (2c_j + x)^2 \end{aligned} \quad (3.16)$$

Note that equation (3.13) can be represented as $f_1(A_1) + g_1(A_1)A_2 = 0$ and equation (3.14) can be represented by $f_2(A_2) + g_2(A_2)A_1 = 0$. Substituting X_j into $f_j(x)$, gives

$$f_j(X_j) = X_j \left[k_j X_j^2 + 4c_j k_j X_j + \left(4c_j^2 k_j + \frac{(2N-1)(2N-3)}{2p_j N^2} - \frac{2c_1 c_2 p_{j'} \gamma_j}{p_j} \right) \right] \quad (3.17)$$

Each condition implies that the right hand side of equation (3.17) is positive. Since $f_j(x)$ is continuous and $f_j(0) < 0$, by the intermediate value theorem, there exists an A_1^* such that $f_j(A_1^*) = 0$, which implies that $(0, X_2)$ and $(A_1^*, 0)$ solve equation (3.13). Similarly, $(X_1, 0)$ and $(0, A_2^*)$ solve equation (4.14). On the other hand, in equation (3.13) and (3.14), a positive root A_j is decreasing as $A_{j'}$ is increasing. Note that $(0, X_2)$ solves equation (3.14). Constructing two convergent sequences. Setting $A_2^* = A_2^1$, substituting A_2^1 into equation (3.13) and solving it, gives the positive A_1^1 . Similarly, substituting A_1^1 into equation (3.14), gives the positive A_2^2 . Repeating this process results in a sequence $\{a_1, b_1, a_2, b_2, \dots, a_n, b_n, \dots\}$, where

$$\begin{aligned} a_n &= (A_1^n, A_2^n), \quad n = 1, 2, 3, \dots \\ b_n &= (A_1^n, A_2^{n+1}), \quad n = 1, 2, 3, \dots \end{aligned}$$

It can be seen that a_n and b_n solve equation (3.13) and (3.14), respectively. Furthermore, $\{A_1^n\}$ is increasing and $\{A_2^n\}$ is decreasing. The reason for this is that

A_j is increasing if and only if $A_{j'}$ is decreasing, and the process of constructing the sequence $\{a_n, b_n\}$ guarantees this. Since A_1^* is an upper bound of $\{A_1^n\}$ and 0 is a lower bound of $\{A_2^n\}$, they are both convergent and the limit solves both equations (3.13) and (3.14). Note that the existence can also be proved by the intermediate value theorem. Now, moving on to prove the uniqueness: let $F_{13}(A_1)$ and $F_{14}(A_1)$ denote the implicit functions that solve equations (3.13) and (3.14), i.e. $(A_1, F_{13}(A_1))$ solves (3.13) for all A_1 and $(A_2, F_{14}(A_1))$ solves (3.14) for all A_1 . Uniqueness would then follow, if it can be shown that there exists at most one A_1 s.t. $F_{13}(A_1) = F_{14}(A_1)$. The latter would hold, if it can be shown that F_{13} is concave and F_{14} is convex. Note that, since $F_{13}(0) = X_2 > F_{14}(0) = A_2^*$, this would also guarantee existence of a pair of solutions, as established in the first part, but this proof of existence is non-constructive. It is now necessary to show that F_{13} is concave and F_{14} is convex. In equation (3.13), A_2 can be represented by a function of A_1 , i.e., $A_2 = -\frac{f_1(A_1)}{g_1(A_1)} = F_{13}(A_1)$. It can be shown that A_2 is strictly decreasing, as the derivative is given by

$$\frac{dA_2}{dA_1} = \frac{-f_1'(A_1)g_1(A_1) + f_1(A_1)g_1'(A_1)}{g_1^2(A_1)}$$

Note that either condition (1) or (2) implies that $f_1(A_1)$ is negative in $[0, A_1^*)$. It can be proved that $-f_1'(A_1)g_1(A_1) + f_1(A_1)g_1'(A_1)$ is decreasing by deriving the derivative,

$$\frac{d}{dA_1} [-f_1'(A_1)g_1(A_1) + f_1(A_1)g_1'(A_1)] = -f_1''(A_1)g_1(A_1) + f_1(A_1)g_1''(A_1) < 0$$

and $g_1(A_1)$ is strictly increasing. This implies that $\frac{dA_2}{dA_1}$ is a strictly decreasing function, which means that the second derivative of $F_{13}(A_1)$ is negative. Therefore, A_2 is a strictly decreasing and strictly concave function of A_1 . Similarly, in equation

(3.14), $A_1 = \frac{f_2(A_2)}{g_2(A_2)}$ is a strictly concave function of A_2 . Since $\frac{f_2(A_2)}{g_2(A_2)}$ is strictly decreasing, the inverse function exists. It can be shown that the inverse function of a strictly convex function is strictly concave by the definition. Therefore, $F_{13}(A_1)$ is strictly concave and $F_{14}(A_1)$ is strictly convex and uniqueness is guaranteed. \square

Proposition 3.1.1 shows that under the given assumptions $V_i(t, x; T)$ has a limit given by $A_1\sqrt{x_1} + A_2\sqrt{x_2} + A_3$, where A_1 , A_2 and A_3 are the positive fixed points of (3.12). Indeed, supposing that

$$\Omega = \{(A_1(t; T), A_2(t; T)) \geq (0, 0) \mid A_1'(t; T) \leq 0, A_2'(t; T) \leq 0\},$$

, it can be seen that Ω is non empty since $(F(t), 0) \in \Omega$ for some functions $F(t) < \frac{2N-1}{2c_1\gamma_2p_1N^3}$ for all $t \in [0, T]$. Note that the first two equations in system (3.10) can be rewritten as

$$\begin{aligned} A_1'(t; T) &= \frac{f_1(A_1(t; T))}{(2c_1 + A_1(t; T))^2} + \frac{\gamma_2 A_2(t; T)}{2}, \\ A_2'(t; T) &= \frac{f_2(A_2(t; T))}{(2c_2 + A_2(t; T))^2} + \frac{\gamma_1 A_1(t; T)}{2}, \end{aligned}$$

where $f_j(x)$ is defined in (3.16). It can be proved that the RHS of the above two equations are increasing in $A_1(t; T)$ and $A_2(t; T)$, respectively. Indeed,

$$\begin{aligned} \frac{d}{dx} \frac{f_j(x)}{(2c_j + x)^2} &= \frac{f_j'(x)(2c_j + x) - 2f_j(x)}{(2c_j + x)^3} > 0, \\ \frac{d}{dx} \frac{\gamma_j x}{2} &= \frac{\gamma_j}{2} > 0, \end{aligned}$$

if a pair of functions is chosen such that $(A_1(0; T), A_2(0; T))$ is in Ω . Since they solve the above system, which is equivalent to system (3.12), and the derivatives are negative at $t = 0$, it can be seen that the derivatives are always negative for all

$t \in [0, T]$ and $A_1(T; T) = A_2(T; T) = 0$. On the other hand, as T tends to infinity, each A_j , $j = 1, 2, 3$, tends to its unique positive fixed point. This can be shown by computing the sign of $f_j(A_j(s)) + g_j(A_j(s))A_{j'}(s)$, where $s = T - t$. Therefore, a stationary symmetric feedback Nash equilibrium of the infinite horizon model for agent i is given by

$$u_i^{j*}(x_j) = \frac{(2N-1)^2 x_j}{p_j N^3 (2c_j + A_j)^2}, \quad j = 1, 2 \quad (3.18)$$

and the corresponding optimal utility function is defined by

$$V_i(x_1, x_2) = A_1 \sqrt{x_1} + A_2 \sqrt{x_2} + A_3. \quad (3.19)$$

Note that in equation (3.18), $u_i^*(0) = 0$. This guarantees that the strategy pair derived is admissible. It can be proved that equation (3.19) is indeed a solution of the Hamilton-Jacobi-Bellman equation for the infinite horizon model. A minor adaptation of a standard verification theorem such as Oksendal, [?] Theorem 11.2.1 implies that the function $V_i(x_1, x_2)$ is in fact the value function for the stochastic differential game. The details are omitted here.

Moving on to the case of a predator-prey system. Without loss of generality, it is assumed that $\alpha_1 < 0$, $\alpha_2 > 0$, $\beta_1 = \beta_2 = 0$, $\gamma_1 < 0$ and $\gamma_2 > 0$, i.e., x_1 is the predator and x_2 is the prey. The following proposition guarantees that equation (3.13) and (3.14) have a unique pair of positive roots in $[0, X_1] \times [A_2^1, X_2]$, where A_2^1 is defined in the proof of Proposition 3.1.1 and X_1 is the unique root for $f_1(A_1)$ defined in system (3.16).

Proposition 3.1.2. (*Predator-Prey Case*) *The system of equations in (3.13) and (3.14) have a unique pair of positive roots in $[0, X] \times [A_2^1, X_2]$ if one of the following*

conditions

$$\begin{aligned} \text{Condition (1).} \quad & 4c_1c_2p_1\gamma_2 \leq \frac{(2N-1)(2N-3)}{N^2} \\ \text{Condition (2).} \quad & X_2 \geq \frac{-2c_2k_2 + \sqrt{\frac{2c_1c_2p_1\gamma_2}{p_2} - \frac{(2N-1)(2N-3)}{2p_2N^2}}}{k_2} \end{aligned}$$

and

$$\text{Condition (3).} \quad 48c_2^2k_2p_2N^3 \geq (2N-1)(2N-3)$$

hold.

Proof. To prove the existence and uniqueness of the pair of positive roots, this analysis takes advantage of system (3.16). Note that either condition (1) or (2) implies that $f_2(A_2)$ has a root at $A_2^1 < X_2$. On the other hand, it has been already shown that $A_2 = -\frac{f_1(A_1)}{g_1(A_1)}$ is decreasing in $[0, X_1]$. Condition (3) leads $A_1 = -\frac{f_2(A_2)}{g_2(A_2)}$ to be a strictly increasing function in $[A_2^1, \infty)$, which therefore implies that the inverse function exists and increases in $[0, \infty)$. Continuing with the notation $F_{13}(A_1)$ and $F_{14}(A_1)$ from the competitive case and it can be seen that $F_{13}(A_1)$ is decreasing and $F_{14}(A_1)$ is increasing in $[0, X_1]$. Moreover, $F_{13}(0) = X_2 > F_{14}(0) = A_2^1$ and $F_{13}(X_1) = 0 < F_{14}(X_1)$. It can be proved via the intermediate value theorem that there exists a pair of positive roots. On the other hand, since $F_{13}(A_1)$ and $F_{14}(A_1)$ are strictly decreasing and increasing respectively, uniqueness is guaranteed. \square

To guarantee that the $(A_1(t; T), A_2(t; T))$ converges to (A_1, A_2) for T tending to infinity, the idea for the competitive case is adopted. Note that it is necessary to start at some points in $[0, X] \times [A_2^1, X_2]$ instead of $[0, X_1] \times [0, X_2]$. The details are omitted here.

3.2. Sensitivity analysis

This section will examine how each of the various parameters affects the optimal utility and control functions. This will start with the case of a two species competitive ecological system. The discussion will be divided into three parts: The first part consists of considering the coefficients which do not relate to equation (3.13) and (3.14); the second part will study those that affect only one of equations (3.13) and (3.14); while the last part consists of considering those which change both equations (3.13) and (3.14). It can be seen that α_j is the only coefficient which is not related to equation (3.13) and (3.14). It is then obvious that A_1 and A_2 do not depend on α_j , but A_3 , however does. In fact, A_3 is increasing as either α_1 or α_2 is increasing. As indicated above, α_j is related to the birth rate of species j and it can be seen that a higher birth rate leads to a larger optimal utility. The coefficients affecting only one of the equations (3.13) and (3.14) are r and N . Taking the following proposition:

Proposition 3.2.1. (*Competitive Case*) *The fixed point A_1 and A_2 are both decreasing in r and N .*

Proof. The concept of this proof is similar to Proposition 3.1.1 and it will use the same notation. Given $r' > r$, we construct sequences B_1^i and B_2^i in analogy to the proof of Proposition 3.1.1. For the r' equation (3.16) has a lower unique positive root, for $j = 1, 2$. Suppose that $(0, B_2^1)$ solves equation (3.14). Then it can be seen that $B_2^1 < A_2^1$. Substituting B_2^1 into equation (3.13) allows derivation of the positive root B_1^1 . Now it can be seen that in equation (3.13), for a fixed A_2 , the positive root A_1 is decreasing as r is increasing: : Differentiating equation (3.13) with respect to

r , gives

$$A_1^3 + 3k_1 A_1^2 \frac{\partial}{\partial r} A_1 + 4c_1 A_1^2 + 8c_1 k_1 A_1 \frac{\partial}{\partial r} A_1 + 4c_1^2 k_1 A_1 + \left[4c_1^2 k_1 + \frac{(2N-1)(2N-3)}{2p_1 N^2} \right] \frac{\partial}{\partial r} A_1 + (\gamma_2 A_1 A_2 + 2c_1 \gamma_2 A_2) \frac{\partial}{\partial r} A_1 = 0 \quad (3.20)$$

Performing the same operation with equation (3.14), but omitting the computation from the resulting two equations, by extracting $\frac{\partial}{\partial r} A_1$ and using condition (1), it can be seen that $\frac{\partial}{\partial r} A_1 < 0$. Substitute B_2^1 into equation (3.13) with r to derive the positive root A_1 and compare it to B_1^1 . It can be seen that B_1^1 is lower than A_1 . Substituting B_1^1 into equation (3.14) and deriving the positive root B_2^2 , it can be shown that B_2^2 is lower than the positive root of equation (3.14) with r and B_1^1 . Repeating this process allows construction of a sequence which is lower than the one we have constructed with r . Furthermore, the sequence satisfies the same properties we have mentioned in the proof of Proposition 3.1.1. Therefore, the limit is lower than the one with r .

Looking at the case of N , it was proved earlier that, with a fixed value of A_2 , in equation (3.13), a larger N implies a lower positive root A_1 and similarly, it can be shown that equation (3.14) has the same property with a fixed A_1 . The derivative of equation (3.13) with respect to N is given by

$$3k_1 A_1^2 \frac{\partial}{\partial N} A_1 + 8c_1 k_1 A_1 \frac{\partial}{\partial N} A_1 + \frac{4N-3}{p_1 N^4} A_1 + \left[4c_1^2 k_1 + \frac{(2N-1)(2N-3)}{2p_1 N^2} \right] \frac{\partial}{\partial N} A_1 + \frac{c_j(4N-3)}{p_j N^4} = 0. \quad (3.21)$$

Similarly, a sequence can be constructed which satisfies all properties in Proposition 3.1.1, but is lower than the one in Proposition 3.1.1. The idea is similar to the case

of r and the proof is omitted. \square

It follows from Proposition 3.2.1 that a larger discount r implies a lower optimal utility. On the other hand, it follows from equation (3.18), that u_i^{j*} is increasing as r increases. The economic interpretation of this is that agents facing a larger discount rate have to harvest more to keep higher payoffs. The impact of the number of agents on individuals is that optimal utility is decreased when N is increased. Furthermore

$$\lim_{N \rightarrow \infty} u_i^{j*}(x_j) = \lim_{N \rightarrow \infty} \frac{(2N-1)^2 x_j}{p_j N^3 (2c_j + A_j)^2} = 0, \quad j = 1, 2.$$

On the other hand, since

$$\begin{aligned} \frac{d}{dN} \frac{(2N-1)^2}{N^3} &= -\frac{(2N-1)(2N-3)}{N^4} < 0, \\ \frac{d}{dN} \frac{(2N-1)^2}{N^2} &= \frac{4N-2}{N^3} > 0, \end{aligned}$$

more agents joining the fishery game implies a greater aggregate amount of harvest of species j , but a lower individual harvest of species j for each agent.

Now, moving on to the second case, i.e., those coefficients which change only one of the equations (3.13) and (3.14). They are c_j , p_j , β_j , γ_j and σ_j for $j = 1, 2$. In the case of c_j , economically, a larger c_j implies a larger cost of x_j and intuitively, agents may harvest less x_j , which implies that A_j is increasing in c_j . However, mathematically this is not correct. Even though the cost of x_j increases, x_j may be still more profitable than x'_j and each agent could still have an incentive to harvest more x_j . The numerical example presented in section 3.5 is a counterexample. The following propositions examine the remaining parameters:

Proposition 3.2.2. (*Competitive Case*) For a fixed $p_{j'}$, A_j is increasing and $A_{j'}$

is decreasing as p_j is decreasing. This result also holds for β_j and σ_j . On the other hand, for a fixed $c_{j'}$ or $\gamma_{j'}$, a larger γ_j leads to a larger A_j and a lower $A_{j'}$.

Proof. Without loss of generality, we suppose that $j = 1$. First considering the case of p_1 . Note that p_1 is not a coefficient in equation (3.14). Multiply equation (3.13) by $2p_1N^3(2c_1 + A_1)^2$ and fix A_2 . The derivative of this equation with respect to p_1 is given by

$$\begin{aligned} & k_1N^3A_1(2c_1 + A_1)^2 + 2k_1p_1N^3A_1(2c_1 + A_1)\frac{\partial}{\partial p_1}A_1 + k_1p_1N^3(2c_1 + A_1)^2\frac{\partial}{\partial p_1}A_1 \\ & + 2N(N-1)(2N-1)\frac{\partial}{\partial p_1}A_1 + \gamma_2N^3A_2(2c_1 + A_1)^2 + 2p_1\gamma_2N^3A_2(2c_1 + A_1)\frac{\partial}{\partial p_1}A_1 = 0. \end{aligned} \quad (3.22)$$

It follows from equation (3.22) that $\frac{\partial}{\partial p_1}A_1 < 0$. By applying the same idea as in the proof of Proposition 3.1.1. For $p'_1 > p_1$, a sequence (B_1^i, B_2^i) is constructed which is compared to the original sequence (A_1^i, A_2^i) . First solve equation (3.16) with $j = 2$ and then derive the positive root A_2^1 . It can be seen that $(0, A_2^1)$ is a solution of equation (3.14). Substituting of A_2^1 into equation (3.13) allows derivation of the positive root B_1^1 . It is lower than A_1^1 derived in Proposition 3.1.1. Using B_1^1 to find the positive root B_2^2 in equation (3.14), it can be seen that B_2^2 is larger than A_2^2 in Proposition 3.1.1. Repeating this process, gives

$$\begin{aligned} B_1^n &< A_1^n, \quad n = 1, 2, \dots \\ B_2^n &> A_2^n, \quad n = 1, 2, \dots \end{aligned}$$

Therefore, the limit of B_1^n for n tending to infinity is lower than A_1 and the limit of B_2^n is larger than A_2 . The same argument also works for the cases of β_j and σ_j .

On the other hand, in the case of γ_j , for fixed γ_2 , equation (3.14) for a fixed A_2

implies that

$$\frac{\partial}{\partial \gamma_1} A_1 = \frac{f_2(A_2) \gamma_1 (2c_2 + A_2)}{g_2^2(A_2)} < 0 \quad (3.23)$$

which therefore implies that A_1 is decreasing as γ_1 is increasing. Starting at $(0, A_2^1)$ which was constructed in the proof in Proposition 3.1.1. Substituting $A_2^1 = B_2^1$ into equation (3.13), the positive root is given by $B_1^1 = A_1^1$. Now substitute B_1^1 into equation (3.14) and the positive root is B_2^2 . Since A_1 is decreasing as γ_1 is increasing, $B_2^2 < a_2^2$ and iteratively $B_2^i < a_2^i$. Similarly, the relationship $B_1^i > A_1^i$ can be obtained. Therefore, the result holds. \square

In the case of p_j , a lower p_j implies a higher price of x_j , see equation (3.5). Agents will then harvest less x_j and more $x_{j'}$ to keep high utility. The coefficients β_j , σ_j and γ_j all have an impact on x_j . It can be seen that larger β_j , σ_j and γ_j lead to a smaller x_j . There are differences between the coefficients, though. In the case of β_j and σ_j , a larger β_j or σ_j leads agents to have more incentives to harvest x_j now, as the biomass of x_j is likely to be less in the future. On the other side, although a higher γ_j implies a lower biomass of x_j , agents can harvest more $x_{j'}$ to keep the stock of x_j at a higher level, which reduces the harvest of x_j and increases the harvest of $x_{j'}$.

Moving on to the case of a predator-prey system.

Proposition 3.2.3. (*Predatory Case*) A_2 is decreasing in either r or N .

Proof. Let $r' > r$ be given and fix A_1 . It can be shown that $f_{1,r'}(A_1) > f_{1,r}(A_1)$, which hence implies that $F_{13,r'}(A_1) = -\frac{f_{1,r'}(A_1)}{g_1(A_1)} < -\frac{f_{1,r}(A_1)}{g_1(A_1)} = F_{13,r}(A_1)$. For any increasing function $H(x)$ defined on $[0, X_1]$ with $H(0) < F_{13,r'}(0)$, where X has been defined in the proof of Proposition 3.1.2, $F_{13,r'}(A_1) - H(A_1) < F_{13,r}(A_1) - H(A_1)$ and $F_{13,r'}(x_{r'}) < F_{13,r}(x_r)$, where $x_{r'}$ and x_r are the roots of $F_{13,r'}(A_1) - H(A_1)$ and $F_{13,r}(A_1) - H(A_1)$ respectively. It can be proved via condition (3) in Proposition 3.1.2 that $F_{14,r'}(A_1) < F_{14,r}(A_1)$. On the other hand, it can also be shown that

for any decreasing function $K(x)$ defined on $[0, X]$ with $K(0) > F_{14,r}(0)$, $K(A_1) - F_{14,r'}(A_1) > K(A_1) - F_{14,r}(A_1)$ as well as $F_{14,r'}(x_{r'}) < F_{14,r}(x_r)$, where $x_{r'}$ and x_r are the roots of $K(A_1) - F_{14,r'}(A_1)$ and $K(A_1) - F_{14,r}(A_1)$ respectively. Since $F_{13,r'}(A_1) - F_{14,r}(A_1) < F_{13,r}(A_1) - F_{14,r}(A_1)$, $A_{2,r} > A_{2,r'}$. Similarly, $F_{13,r'}(A_1) - F_{14,r'}(A_1) > F_{13,r'}(A_1) - F_{14,r}(A_1)$ implies that $A_{2,r,r'} > A_{2,r'}$. Therefore, A_2 is decreasing in r . In the case of N , the proof is analogous to the case of r and it is omitted here. \square

Proposition 3.2.3 states that either a larger discount rate r or a larger number of agents N implies more harvest of x_2 , the prey. Since x_2 is the food source of x_1 , it reduces the biomass of x_1 , which in turn increases the costs of each agent harvesting x_1 . As a result, agents may have less incentive to harvest larger amounts of x_1 . On the other hand, mathematically, there exist two groups of parameters such that one leads A_1 to increase in either r or N and the other leads to the opposite. For example, suppose that $r = 0.2$, $\alpha_1 = -1.3$, $\alpha_2 = 4$, $\beta_1 = 0.25$, $\beta_2 = 0.1$, $\gamma_1 = -1$, $\gamma_2 = 0.7$, $\sigma_1 = 0.15$, $\sigma_2 = 0.55$, $p_1 = 0.3$, $p_2 = 1$, $c_1 = 0.45$, $c_2 = 1$ and $N = 2$, it can be derived numerically that $A_1 = 0.21356$ and $A_2 = 0.48059$. On the other hand, if $r = 0.5$ while all other parameters remain the same, then $A_1 = 0.21753$ and $A_2 = 0.2746$. Note that in this case, x_1 is more profitable and costly than x_2 . Even though agents expect that the food resource of x_1 , i.e., x_2 , is reduced, they still harvest less x_1 and therefore A_1 decreases. This result also holds for N ; even though more agents join the game, each agent still harvest less x_1 . On the other hand, in the case of σ_j , γ_j and p_j , the following proposition can be put forward:

Proposition 3.2.4. (*Predatory Case*) A_1 and A_2 are both decreasing in γ_2 as well as A_1 and A_2 is decreasing and increasing in γ_1 , respectively. In the case of σ_j , for a fixed σ_2 , A_1 and A_2 are decreasing in σ_1 . On the other hand, for a fixed σ_1 , A_1 is increasing and A_2 is decreasing in σ_2 . In the case of p_1 , A_1 and A_2 are decreasing

in p_1 .

Proof. The proof of this result is analogous to the proof of Proposition 3.2.3. \square

In the case of σ_j , a larger σ_1 implies a greater impact of x_1 in the future and therefore each agent tends to harvest more x_1 . Since less x_1 leads the biomass of x_2 to increase, agents will harvest more x_2 simultaneously. On the other hand, an increasing σ_2 leads agents to harvest more x_2 and leads x_1 to have a lesser resource of food. Hence agents tend to harvest less x_1 . In the case of γ_1 , a larger $-\gamma_1$ leads x_1 to increase and then x_2 is decreasing due to an increased stock of predators. It is interesting that this result shows that fishery agents harvest less x_1 and more x_2 . The interpretation is that agents may observe a lower biomass of x_2 in the future and hence catch more x_2 now. This leads x_1 to decrease in the future due to a lack of food resources and therefore agents harvest less x_1 . On the other side, a larger γ_2 causes x_2 to decrease, while x_1 is decreasing since it is the predator. Agents expect lower stock of both species in the future and therefore, they catch more x_1 and x_2 now. In the case of p_1 , a higher p_1 leads agents to benefit less from x_1 . They will need to harvest more x_1 to maintain the same level of utility. On the other hand, this leads the biomass of x_2 to increase and they also tend to harvest more x_2 . In the case of c_j and p_2 , however, it is not possible to give a definitive answer as to how agents change their strategies and how those parameters affect the fixed points are not clear. It can also be seen that the harvest of x_2 , the prey, is increasing in each parameter, which may cause x_1 to decrease in the future. The results of this section have been organised into the following tables, where the first and second tables represent the competitive and predatory-prey cases, respectively.

Parameters	Fixed Points
r or N	A_1 and A_2 are decreasing.
p_j, β_j or σ_j	A_j is decreasing and $A_{j'}$ is increasing.
c_j or γ_j	A_j is increasing and $A_{j'}$ is decreasing.

TABLE 3.1: Competitive Case

Parameters	Fixed Points
r or N	A_2 is decreasing.
γ_1	A_1 is decreasing and A_2 is increasing.
p_1 or γ_2	A_1 and A_2 are decreasing.
σ_1	A_1 and A_2 are decreasing.
σ_2	A_1 is increasing and A_2 is decreasing.

TABLE 3.2: Predator-Prey Case

3.3. The case of single-species restricted fisheries

In this section, without loss of generality, it is supposed that there are two agents, say agent 1 and agent 2, and agent i is only allowed to harvest species x_i . This assumption is realistic given that fishery agents could harvest via different fishing vessels and techniques. Datta and Mirman (1999) also considered this case and argue that this situation could occur, see [?]. In [?], the fishery agents represent different countries which have different consumption characteristics and species preferences. The objective functional for i is defined by

$$\max_{u_i} E \left\{ \int_0^\infty e^{-rt} \left[\frac{\sqrt{u_i(t)}}{\sqrt{p_i}} - \frac{c_i u_i(t)}{\sqrt{x_i(t)}} \right] dt \mid x_1(0) = x_{10}, x_2(0) = x_{20} \right\}, \quad (3.24)$$

and state equations are given by

$$\begin{aligned} dx_i(t) &= \left\{ x_i(t) \left[\frac{\alpha_i}{\sqrt{x_i(t)}} - \beta_i - \gamma_i \sqrt{\frac{x_{i'}(t)}{x_i(t)}} \right] - u_i(t) \right\} dt + \sigma_i x_i(t) dW_i(t), \\ dx_{i'}(t) &= \left\{ x_{i'}(t) \left[\frac{\alpha_{i'}}{\sqrt{x_{i'}(t)}} - \beta_{i'} - \gamma_{i'} \sqrt{\frac{x_i(t)}{x_{i'}(t)}} \right] - u_{i'}(t) \right\} dt + \sigma_{i'} x_{i'}(t) dW_{i'}(t). \end{aligned} \quad (3.25)$$

As before, the finite horizon approximation approach will be applied to solve this problem. The Hamilton-Jacobi-Bellman equations for the finite time horizon approximation for agents $i = 1, 2$ is given by

$$\begin{aligned} & rV_1(t, x_1, x_2; T) - \frac{\partial}{\partial t} V_1(t, x_1, x_2; T) \\ &= \max_{u_1} \left\{ \frac{\sqrt{u_1}}{\sqrt{p_1}} - \frac{c_1 u_1}{\sqrt{x_1}} + \frac{\partial}{\partial x_1} V_1(t, x_1, x_2; T) (\alpha_1 \sqrt{x_1} - \beta_1 x_1 - \gamma_1 \sqrt{x_1 x_2} - u_1) \right. \\ &\quad + \frac{\partial}{\partial x_2} V_1(t, x_1, x_2; T) (\alpha_2 \sqrt{x_2} - \beta_2 x_2 - \gamma_2 \sqrt{x_1 x_2} - u_2^*) \\ &\quad \left. + \frac{\sigma_1^2 x_1^2}{2} \frac{\partial^2}{\partial x_1^2} V_1(t, x_1, x_2; T) + \frac{\sigma_2^2 x_2^2}{2} \frac{\partial^2}{\partial x_2^2} V_1(t, x_1, x_2; T) \right\} \end{aligned} \quad (3.26)$$

and

$$\begin{aligned} & rV_2(t, x_1, x_2; T) - \frac{\partial}{\partial t} V_2(t, x_1, x_2; T) \\ &= \max_{u_2} \left\{ \frac{\sqrt{u_2}}{\sqrt{p_2}} - \frac{c_2 u_2}{\sqrt{x_2}} + \frac{\partial}{\partial x_2} V_2(t, x_1, x_2; T) (\alpha_2 \sqrt{x_2} - \beta_2 x_2 - \gamma_2 \sqrt{x_1 x_2} - u_2) \right. \\ &\quad + \frac{\partial}{\partial x_1} V_2(t, x_1, x_2; T) (\alpha_1 \sqrt{x_1} - \beta_1 x_1 - \gamma_1 \sqrt{x_1 x_2} - u_1^*) \\ &\quad \left. + \frac{\sigma_1^2 x_1^2}{2} \frac{\partial^2}{\partial x_1^2} V_2(t, x_1, x_2; T) + \frac{\sigma_2^2 x_2^2}{2} \frac{\partial^2}{\partial x_2^2} V_2(t, x_1, x_2; T) \right\}. \end{aligned} \quad (3.27)$$

Necessary conditions for maximizing the RHS of equation (3.26) and (3.27) are

$$u_i = \frac{1}{4p_i} \frac{1}{\left[\frac{c_i}{\sqrt{x_i}} + \frac{\partial}{\partial x_i} V_i(t, x_i, x_{i'}, T) \right]^2} \quad (3.28)$$

for $(i, i') = (1, 2)$ and $(2, 1)$. Substituting equation (4.28) into equation (4.26) and (4.27), results in a system of PDE's

$$\begin{aligned} & rV_1(t, x_1, x_2; T) - \frac{\partial}{\partial t} V_1(t, x_1, x_2; T) \\ &= \frac{1}{2p_1 \left[\frac{c_1}{\sqrt{x_1}} + \frac{\partial}{\partial x_1} V_1(t, x_1, x_2; T) \right]} - \frac{c_1}{4p_1 \sqrt{x_1} \left[\frac{c_1}{\sqrt{x_1}} + \frac{\partial}{\partial x_1} V_1(t, x_1, x_2; T) \right]^2} \\ &+ \frac{\partial}{\partial x_1} V_1(t, x_1, x_2; T) \left\{ \alpha_1 \sqrt{x_1} - \beta_1 x_1 - \gamma_1 \sqrt{x_1 x_2} - \frac{1}{4p_1 \left[\frac{c_1}{\sqrt{x_1}} + \frac{\partial}{\partial x_1} V_1(t, x_1, x_2; T) \right]^2} \right\} \\ &+ \frac{\partial}{\partial x_2} V_1(t, x_1, x_2; T) \left\{ \alpha_2 \sqrt{x_2} - \beta_2 x_2 - \gamma_2 \sqrt{x_1 x_2} - \frac{1}{4p_2 \left[\frac{c_2}{\sqrt{x_2}} + \frac{\partial}{\partial x_2} V_2(t, x_1, x_2; T) \right]^2} \right\} \\ &+ \frac{\sigma_1^2 x_1^2}{2} \frac{\partial^2}{\partial x_1^2} V_1(t, x_1, x_2; T) + \frac{\sigma_2^2 x_2^2}{2} \frac{\partial^2}{\partial x_2^2} V_1(t, x_1, x_2; T) \end{aligned} \quad (3.29)$$

and

$$\begin{aligned}
& rV_2(t, x_1, x_2; T) - \frac{\partial}{\partial t}V_2(t, x_1, x_2; T) \\
&= \frac{1}{2p_2 \left[\frac{c_2}{\sqrt{x_2}} + \frac{\partial}{\partial x_2}V_2(t, x_1, x_2; T) \right]} - \frac{c_2}{4p_2\sqrt{x_2} \left[\frac{c_2}{\sqrt{x_2}} + \frac{\partial}{\partial x_2}V_2(t, x_1, x_2; T) \right]^2} \\
&+ \frac{\partial}{\partial x_2}V_2(t, x_1, x_2; T) \left\{ \alpha_2\sqrt{x_2} - \beta_2x_2 - \gamma_2\sqrt{x_1x_2} - \frac{1}{4p_2 \left[\frac{c_2}{\sqrt{x_2}} + \frac{\partial}{\partial x_2}V_2(t, x_1, x_2; T) \right]^2} \right\} \\
&+ \frac{\partial}{\partial x_1}V_2(t, x_1, x_2; T) \left\{ \alpha_1\sqrt{x_1} - \beta_1x_1 - \gamma_1\sqrt{x_1x_2} - \frac{1}{4p_1 \left[\frac{c_1}{\sqrt{x_1}} + \frac{\partial}{\partial x_1}V_1(t, x_1, x_2; T) \right]^2} \right\} \\
&+ \frac{\sigma_1^2 x_1^2}{2} \frac{\partial^2}{\partial x_1^2}V_2(t, x_1, x_2; T) + \frac{\sigma_2^2 x_2^2}{2} \frac{\partial^2}{\partial x_2^2}V_2(t, x_1, x_2; T)
\end{aligned} \tag{3.30}$$

with boundary conditions

$$V_1(T, x_1, x_2; T) = 0 \tag{3.31}$$

$$V_2(T, x_1, x_2; T) = 0 \tag{3.32}$$

As before, a sophisticated guess will be made with regard to the functional form of the solution for equations (3.29), (3.30), (3.31) and (3.32)

$$V_1(t, x_1, x_2; T) = A_1(t)\sqrt{x_1} + A_2(t)\sqrt{x_2} + A_3(t)$$

$$V_2(t, x_1, x_2; T) = B_1(t)\sqrt{x_1} + B_2(t)\sqrt{x_2} + B_3(t).$$

Substituting these into equations (3.29) and (3.30) generates a system of ordinary differential equations. To apply the finite horizon approximation approach, it is then necessary to derive the fixed points and to solve the resulting system of polynomials.

The polynomial system is

$$\begin{aligned}
k_1 A_1^3 + \left(4c_1 k_1 + \frac{\gamma_2 A_2}{2}\right) A_1^2 + \left(4c_1^2 k_1 - \frac{1}{2p_1} + 2c_1 \gamma_2 A_2\right) A_1 + 2c_1^2 \gamma_2 A_2 - \frac{c_1}{p_1} &= 0 \\
\left[k_2 + \frac{1}{2p_2 (2c_2 + B_2)^2}\right] A_2 + \frac{\gamma_1 A_1}{2} &= 0 \\
r A_3 - \frac{\alpha_1 A_1}{2} - \frac{\alpha_2 A_2}{2} &= 0 \\
\left[k_1 + \frac{1}{2p_1 (2c_1 + A_1)^2}\right] B_1 + \frac{\gamma_2 B_2}{2} &= 0 \\
k_2 B_2^3 + \left(4c_2 k_2 + \frac{\gamma_1 B_1}{2}\right) B_2^2 + \left(4c_2^2 k_2 - \frac{1}{2p_2} + 2c_2 \gamma_1 B_1\right) B_2 + 2c_2^2 \gamma_1 B_1 - \frac{c_2}{p_2} &= 0 \\
r B_3 - \frac{\alpha_1 B_1}{2} - \frac{\alpha_2 B_2}{2} &= 0
\end{aligned} \tag{3.33}$$

Economically, in the case of a two species competitive system, it would be expected that A_1 and B_2 will be positive and A_2 and B_1 will be negative. The reason for this is that agent i only benefits from species x_i . On the other hand, a greater biomass of $x_{i'}$ implies a lower biomass of x_i . In the case of a predator-prey system, both A_1 and A_2 are positive, while B_1 is negative and B_2 is positive. The interpretation of this is that an increase in the biomass of x_2 leads the biomass of x_1 to increase, which therefore implies that agent 1 will also benefit from x_2 . In system (3.33), the second equation is substituted into the first equation and the third equation into the fourth, with the following results:

$$\left[k_1 + \frac{\gamma_2 f(B_2)}{2}\right] A_1^3 + [4c_1 k_1 + 2c_1 \gamma_2 f(B_2)] A_1^2 + \left[4c_1^2 k_1 - \frac{1}{2p_1} + 2c_1^2 \gamma_2 f(B_2)\right] A_1 - \frac{c_1}{p_1} = 0 \tag{3.34}$$

and

$$\left[k_2 + \frac{\gamma_1 g(A_1)}{2} \right] B_2^3 + [4c_2 k_2 + 2c_2 \gamma_1 g(A_1)] B_2^2 + \left[4c_2^2 k_2 - \frac{1}{2p_2} + 2c_2^2 \gamma_1 g(A_1) \right] B_2 - \frac{c_2}{p_2} = 0 \quad (3.35)$$

where

$$\begin{aligned} f(B_2) &= -\frac{\gamma_1 p_2 (2c_2 + B_2)^2}{2k_2 p_2 (2c_2 + B_2)^2 + 1} \\ g(A_1) &= -\frac{\gamma_2 p_1 (2c_1 + A_1)^2}{2k_1 p_1 (2c_1 + A_1)^2 + 1} \end{aligned}$$

This gives rise to the following proposition:

Proposition 3.3.1. *Under the assumption that*

$$4c_j^2 k_1 k_2 p_j - c_j^2 p_j \gamma_1 \gamma_2 - 2k_{j'} \geq 0$$

hold for $(j, j') = (1, 2)$ and $(2, 1)$, equation (3.34) and (3.35) have a unique positive pair of solutions.

Proof. The condition with $j = 1$ implies that equation (3.34) has a positive and unique root. Similarly, when $j = 2$, the condition leads equation (3.35) to have a positive and unique root. Note that, in the case of a two species competitive system, it is possible to construct two increasing sequences and each sequence has an upper boundary due to this condition. On the other hand, in the case of a predator-prey system, one of these two sequences will increase with an upper bound while the other will decrease with a lower bound. Hence, the proof of this result is analogous to the proof of Proposition 3.1.1. \square

The optimal utility and optimal control obtained in the setup of this section will now be compared with the optimal utility from the previous section. Starting with

the case of a two species competitive system: Equations (3.13) and (3.34), consist of cubic polynomials. Moreover, the coefficients of A_1^3 , A_1^2 and A_1 in equation (3.13) are greater than those in equation (3.34). On the other hand,

$$0 > 2c_1^2\gamma_2A_2 - \frac{3c_1}{8p_1} > 2c_1^2\gamma_2A_2 - \frac{c_1}{p_1} > -\frac{c_1}{p_1} \quad (3.36)$$

Similar properties hold for equations (3.14) and (3.35). It can be proved that the solutions for equation (3.13) and (3.14) are less in value than the solutions of equation (3.34) and (3.35). On the other hand, the optimal controls for agent 1 and 2 with restriction are given by

$$\begin{aligned} u_1^{r*} &= \frac{x_1}{p_1(2c_1 + A_1^r)^2} \\ u_2^{r*} &= \frac{x_2}{p_2(2c_2 + B_2^r)^2} \end{aligned}$$

while without any restriction they are

$$u_i^{nr*} = \frac{9x_i}{8p_i(2c_i + A_i^{nr})^2}, \quad i = 1, 2.$$

Since $A_1^{nr} \leq A_1^r$ and $A_2^{nr} \leq B_2^r$, agent i with restriction harvests less than without restriction. The interpretation of this is that overexploitation of x_i leads $x_{i'}$ to increase in biomass. This will therefore reduce the biomass of x_i and increase the utility of agent i' . On the other hand, without any restriction, agents harvest both species and do not worry about the biomass of $x_{i'}$. This may cause one of the species to become extinct. On the other hand, it can be seen that if the following inequalities hold

$$\frac{A_1^r - A_1^{nr}}{A_2^{nr} - A_2^r} \sqrt{x_1} + \frac{A_3^r - A_3^{nr}}{A_2^{nr} - A_2^r} \geq \sqrt{x_2} \geq \frac{B_1^r - A_1^{nr}}{A_2^{nr} - B_2^r} \sqrt{x_1} + \frac{B_3^r - A_3^{nr}}{A_2^{nr} - B_2^r}$$

then agents obtain more utility in the case of restriction than the case of competition.

In the case of a predator-prey system, however, the idea of comparing each coefficient in equation (3.13) and (3.14) to (3.34) and (3.35) cannot be applied. Economically, it would be expected that agent 2 would prefer to harvest more x_2 . The reason for this is that the biomass of x_1 is increasing if agent 2 harvests less x_2 , which therefore implies that agent 1 benefits more from x_1 . On the other hand, even though agent 2 harvests less x_2 to maintain a sustainable biomass of x_2 , x_1 still hunts for x_2 , causing a reduction in the biomass of x_2 . In this case, agent 2 not only competes with agent 1 but also species x_1 . The situation is that if agent 2 harvests more x_2 and causes x_2 become extinct, then x_1 will become extinct in the future. This causes agent 2 to have more bargaining power and agent 1 may be forced to cooperate.

3.4. The case of maximizing joint utility

In this section, it is proposed that all agents are allowed to cooperate. The idea is then to derive the optimal joint utility function. Since the model developed above is symmetric, it is natural to assume that each agent shares the catch equally among all members. The joint objective functional is given by

$$\max_{v^1, v^2} E \left\{ \int_0^\infty e^{-rt} \sum_{j=1}^2 \left[\frac{\sqrt{v^j(t)}}{\sqrt{p_j}} - \frac{c_j}{\sqrt{x_j}} v^j(t) \right] dt \mid x_1(0) = x_{10}, x_2(0) = x_{20} \right\} \quad (3.37)$$

where

$$v^j(t) = \sum_{i=1}^N u_i^j(t), \quad j = 1, 2$$

and the state equations are given by

$$\begin{aligned} dx_1(t) &= \left\{ x_1(t) \left[\frac{\alpha_1}{\sqrt{x_1(t)}} - \beta_1 - \gamma_1 \sqrt{\frac{x_2(t)}{x_1(t)}} \right] - v^1(t) \right\} dt + \sigma_1 x_1(t) dW_1(t) \\ dx_2(t) &= \left\{ x_2(t) \left[\frac{\alpha_2}{\sqrt{x_2(t)}} - \beta_2 - \gamma_2 \sqrt{\frac{x_1(t)}{x_2(t)}} \right] - v^2(t) \right\} dt + \sigma_2 x_2(t) dW_2(t). \end{aligned} \quad (3.38)$$

It can be seen that the model is consistent with the case $N = 1$. Therefore, the optimal controls and joint utility function can easily be derived from equation (3.18) and (3.19). Furthermore, the optimal controls can be obtained by Proposition 3.2.1. On the other hand, the aggregate amount of harvest for the case of cooperation is less than that for the case of competition. This gives rise to a new proposition:

Proposition 3.4.1. *In the case of a two species competitive system, if either condition (1) or (2) in Proposition 3.1.1 and the condition in Proposition 3.3.1 hold, then equation (3.13) and (3.14) under $N = 1$ has a pair of positive and unique fixed points. On the other hand, in the case of a predator-prey system, the conditions in Proposition 3.1.2 and Proposition 3.3.1 guarantees that equation (3.13) and (3.14) under $N = 1$ has a pair of positive roots.*

Proof. The proof of this result is analogous to the proof of Proposition 3.1.1 and Proposition 3.1.2. □

Suppose that (A_1^c, A_2^c) denotes the fixed points for the case of cooperation and (A_1^{nc}, A_2^{nc}) denotes the fixed point for the case of competition. This gives

$$\frac{x_j}{p_j (2c_j + A_j^c)^2} \leq \frac{x_j}{p_j (2c_j + A_j^{nc})^2} \leq N \frac{(2N - 1)^2 x_j}{p_j N^3 (2c_j + A_j^{nc})^2} \quad (3.39)$$

since $\frac{(2N-1)^2}{N^2} \geq 1$. Clark (2006) highlighted several methods to reduce the impact on the ecological system and one of these is cooperation between agents, see [?], Chapter 1. It can be seen in inequality (3.39) that, in the case of cooperation, the impact on the ecological system is less than under competition, i.e., the aggregate amount of harvest under cooperation is lower than under competition. The next section shows an example where this relationship can be reversed, if the system is predator-prey, rather than competitive.

3.5. Some numerical results

This section will present some numerical results from the equations generated above. An iterative method will be applied in order to derive a positive pair of solutions to equation (3.13) and (3.14), as well as equation (3.34) and (3.35). The concept is to start with a sufficiently large discretization of the set of potential values for A_1 i.e. $\{A_1^i | i = 0, \dots, n\}$. Substituting each A_1^i into equation (3.14) and solving it numerically by Newton's method gives $\{A_2^i | i = 0, \dots, n\}$. Each pair (A_1^i, A_2^i) is then examined via equation (3.13) in order to identify which A_1^m leads equation (3.13) to have a minimum. The pair (A_1^m, A_2^m) are then chosen as an approximation of the positive pair of solutions. Note that in the proof of Proposition 4.1.1, Proposition 4.3.1 and Proposition 3.4.1, it can be seen that this technique will leads A_1^i and A_2^i to be consistent with the property of the sequences we have constructed in those proofs.

Starting with the case of a two species competitive system and assuming that the coefficients for the ecological system are given by $\alpha_1 = 1.3$, $\alpha_2 = 1$, $\beta_1 = 0.6$, $\beta_2 = 0.55$, $\gamma_1 = 0.9$, $\gamma_2 = 0.7$, $\sigma_1 = 0.15$ and $\sigma_2 = 0.55$. For the agents it is assumed that $r = 0.2$, $p_1 = 1.85$, $p_2 = 1.65$, $c_1 = 1.45$, $c_2 = 1.3$ and $N = 2$. This case results in Figures 3.3 to 3.12 in Graphical Illustration. Note that

these parameters satisfy Condition (1) in Proposition 3.1.1 and the condition in Proposition 3.3.1. This guarantees that fixed points exist. Under the coefficients chosen, $A_1^{nr} = A_1^{nc} \approx 0.030531$, $A_2^{nr} = A_2^{nc} \approx 0.051864$, $A_3^{nr} = A_3^{nc} \approx 0.22889$, $A_1^r \approx 0.38468$, $A_2^r \approx -0.31754$, $A_3^r \approx 0.45635$, $B_1^r \approx -0.306$, $B_2^r \approx 0.4615$, $B_3^r \approx 0.15926$, $A_1^c \approx 0.078766$, $A_2^c \approx 0.14608$ and $A_3^c \approx 0.62119$. Figure 3.3 represents the optimal utility function without any restriction for each agent. It can be seen that optimal utility is increasing as either x_1 or x_2 is increasing. This is reasonable because agents benefit from both species. On the other hand, in Figure 3.4, agent 1 is bound to harvest x_2 and since x_2 competes with x_1 , a larger x_2 implies a lower optimal utility. Similarly, in Figure 3.5, agent 2 receives less utility if x_1 is larger. The optimal joint utility function is presented in Figure 3.6 while Figure 3.7 displays the difference between the optimal joint utility function and the sum of the utility functions in competition. From the analysis in section 5, if agents observe that the biomass of each species satisfies

$$0.9585\sqrt{x_1} - 0.61575 \geq \sqrt{x_2} \geq 0.82154\sqrt{x_1} - 0.16998,$$

they may prefer the restricted case because they obtain more benefits than under the competitive regime. Figure 3.8 and 3.9 represent the difference of the optimal utility functions between restriction and competition for agent 1 and 2 respectively. As an approximation to an infinite time horizon, this uses a finite time horizon of $T = 50$. The initial states are given by the fixed points of the deterministic ecological system, see section 3.1. In this case, they are $x_{10} = 0.38028$ and $x_{20} = 1.0678$. In the case of no restriction, it can be seen that in Figure 3.10, x_1 is decreasing around x_{10} and then increasing until it converges to around 2.89. On the other hand, x_2 is decreasing and tends to 0.09, which means that x_2 may become extinct relatively easily. The interpretation of this is that since x_2 is more profitable and costs less than x_1 , the

likelihood of x_2 becoming extinct is quite large. Since x_2 cannot compete with x_1 , the biomass of x_1 is increasing. On the other hand, if each agent is restricted, it can be observed from Figure 3.11 that the expectations of both species are increasing and converge to around 3.35 and 0.35, respectively. In the case of the optimal joint utility function, both species survive. It can be seen in Figure 3.12 that x_1 converges to around 3.30 and x_2 tends to around 0.30. In order to prevent over-exploitation of resources, some restriction may be necessary. In this case, either forcing fisheries to fish a single species or cooperation between the fisheries leads both species to maintain a sustainable biomass.

Now moving to the case of a predator prey system: Assuming that $\alpha_1 = -1.3$, $\alpha_2 = 4$, $\gamma_1 = -0.9$, $\gamma_2 = 0.7$, $\sigma_1 = 0.15$ and $\sigma_2 = 0.55$. For the agents it is assumed that $r = 0.2$, $p_1 = 1.85$, $p_2 = 3$, $c_1 = 1.45$, $c_2 = 2$ and $N = 2$. Figures 3.13 to 3.22 in the Graphical Illustration relate to this case. It can be computed numerically that $A_1^{nr} = A_1^{nc} \approx 0.015258$, $A_2^{nr} = A_2^{nc} \approx 0.08893$, $A_3^{nr} = A_3^{nc} \approx 0.83971$, $A_1^r \approx 0.10726$, $A_2^r \approx 0.19463$, $A_3^r \approx 1.5977$, $B_1^r \approx -0.067684$, $B_2^r \approx 0.045$, $B_3^r \approx 0.66997$, $A_1^c \approx 0.039422$, $A_2^c \approx 0.23986$ and $A_3^c \approx 2.2705$. Note that x_1 is more profitable and less costly than x_2 due to those parameters. For instance, if sharks are harvested for their fins this causes the population of sharks to decline. On the other hand, the biomass of other species hunted by sharks increases because the biomass of sharks declines. This clearly has a great effect on the ecological system. In Figure 3.13, it can be seen that even though the parameters selected cause x_1 to be more profitable and less costly than x_2 , each agent benefits more from x_2 than x_1 . This numerical example shows that the benefit from the profitable predator might not be as high as expected. On the other hand, in Figure 3.14, agent 1 obtains higher utility than without any restriction. The difference between the two utility functions is presented in Figure 3.18. The utility function in the case where agent 2 is allowed to harvest only x_2 is

presented in Figure 3.15. It can be seen in Figure 3.19 that agent 2 obtains lower utility than under the unrestricted competition condition. This may lead agent 2 to have less incentive to be restricted. The case of cooperation is presented in figure 3.16 and each agent obtains higher utility than under competitive conditions, which was expected, see Figure 3.17. Note that a larger x_1 in cooperation than under competition.

The graphs relating to the ecological system for each case are presented in Figures 3.20, 3.21 and 3.22. As initial values used are taken from the fixed points of the deterministic ecological system, in this numerical example they are given by $x_{10} = 32.653$ and $x_{20} = 2.0864$. We choose $T = 500$ was chosen for each case. In Figure 3.20, 3.21 and 3.22, it can be seen that x_1 is decreasing and x_2 is increasing at the beginning. Since x_1 has more resources of food, it is increasing and therefore causes x_2 to decrease by predation. However, they are both decreasing around $t = 30$ and both tend to 0, which means that both x_1 and x_2 become extinct. Although Figures 3.20, 3.21 and 3.22 are similar, the rate of convergence to zero is different. In the case of competition, x_1 and x_2 converge to zero with a significantly lower rate. It can be seen that both species survive longer when agents compete with each other. Once x_2 becomes extinct, x_1 will soon be extinct. It can be seen that both cooperation and restriction are not efficient in a predator-prey system and cause both species to become extinct earlier than in the case of competition.

3.6. Conclusions

The author has presented a continuous time game theoretic model of a two species fishery with ecological interaction and uncertainty. This line of research extends and combines previous studies of two species fishery models in discrete time with no uncertainty by Andereson (1975) [?] and Quirk and Smith (1977) [?]

as well as the line of stochastic differential game models developed by Hofbauer and Sigmund (1998) [?] and Jorgensen and Yeung (1996) [?] which focuses on one species fisheries. There is sufficient motivation for either of the two research lines, but it is only by combining them that a more realistic model containing uncertainty, continuous time and ecological aspects can be obtained. The new model is richer, but also mathematically more complex. It has been shown that, under appropriate assumptions, this model has a stationary feedback Nash equilibrium and various formulas and equations have been derived which characterize it. It was possible to determine a semi- analytic solution, by which it is meant that an analytic form of the solution has been derived, but it is necessary to numerically calculate certain parameters within it. The comparative statics of the model have been discussed, together with the numerical results, and an economic interpretation has been given. The different cases of competitive, restricted and cooperative fisheries management have been discussed, along with the differences in their impacts on the ecological system. According to the numerical example, it can be seen from the perspective of conservation that it is necessary to introduce some fishery policies. Therefore, the next chapter will introduce the concept of maximum sustainable yield.

Graphical Illustration

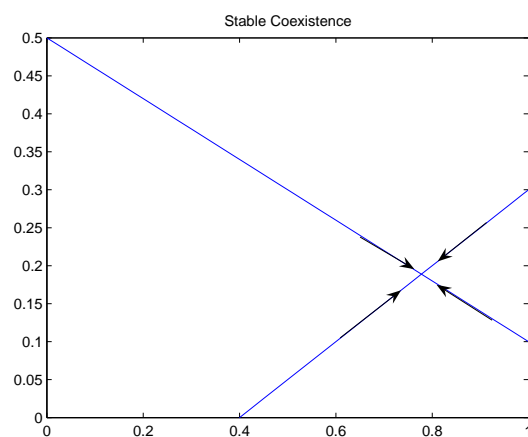


FIGURE 3.1: Stable Coexistence Case

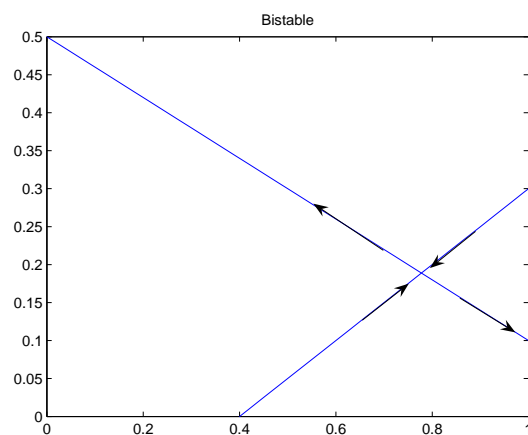


FIGURE 3.2: Bistable Case

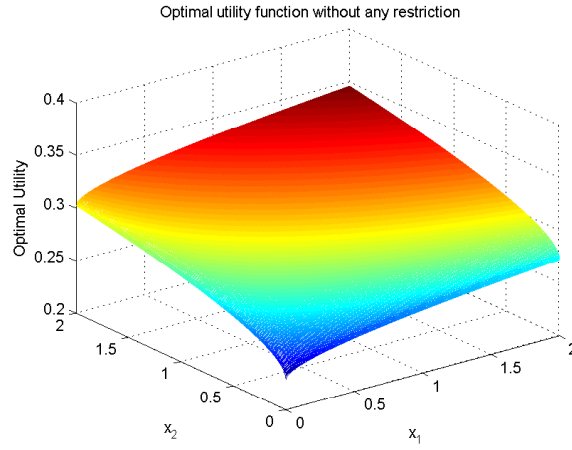


FIGURE 3.3: Optimal utility function in competition under $\alpha_1 = 1.3$, $\alpha_2 = 1$, $\beta_1 = 0.6$, $\beta_2 = 0.55$, $\gamma_1 = 0.9$, $\gamma_2 = 0.7$, $\sigma_1 = 0.15$, $\sigma_2 = 0.55$, $r = 0.2$, $p_1 = 1.85$, $p_2 = 1.65$, $c_1 = 1.45$, $c_2 = 1.3$ and $N = 2$

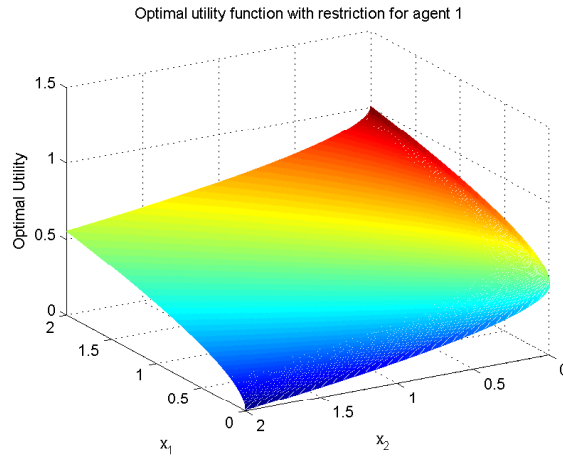


FIGURE 3.4: Optimal utility function with restriction for agent 1 under $\alpha_1 = 1.3$, $\alpha_2 = 1$, $\beta_1 = 0.6$, $\beta_2 = 0.55$, $\gamma_1 = 0.9$, $\gamma_2 = 0.7$, $\sigma_1 = 0.15$, $\sigma_2 = 0.55$, $r = 0.2$, $p_1 = 1.85$, $p_2 = 1.65$, $c_1 = 1.45$ and $c_2 = 1.3$

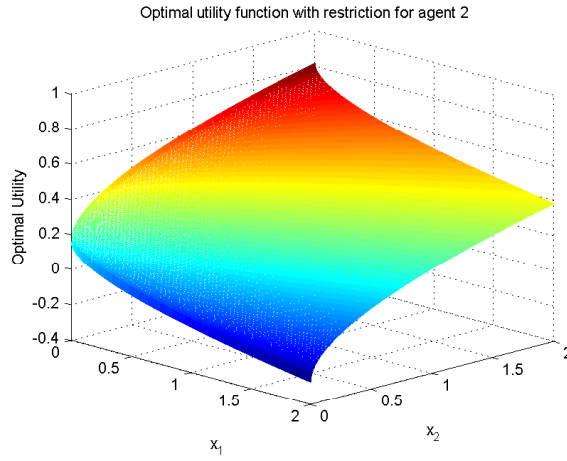


FIGURE 3.5: Optimal utility function with restriction for agent 2 under $\alpha_1 = 1.3$, $\alpha_2 = 1$, $\beta_1 = 0.6$, $\beta_2 = 0.55$, $\gamma_1 = 0.9$, $\gamma_2 = 0.7$, $\sigma_1 = 0.15$, $\sigma_2 = 0.55$, $r = 0.2$, $p_1 = 1.85$, $p_2 = 1.65$, $c_1 = 1.45$ and $c_2 = 1.3$

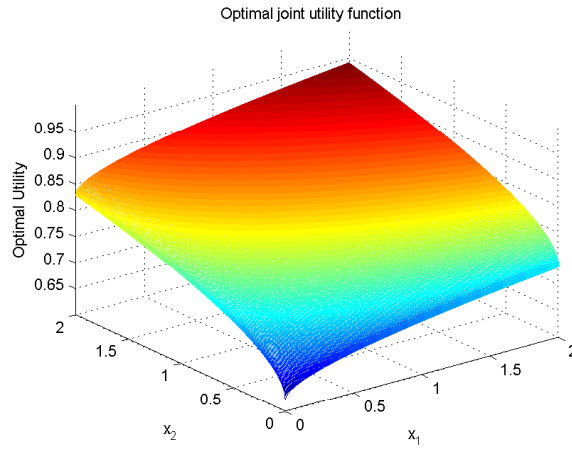


FIGURE 3.6: Optimal joint utility function under $\alpha_1 = 1.3$, $\alpha_2 = 1$, $\beta_1 = 0.6$, $\beta_2 = 0.55$, $\gamma_1 = 0.9$, $\gamma_2 = 0.7$, $\sigma_1 = 0.15$, $\sigma_2 = 0.55$, $r = 0.2$, $p_1 = 1.85$, $p_2 = 1.65$, $c_1 = 1.45$, $c_2 = 1.3$ and $N = 2$

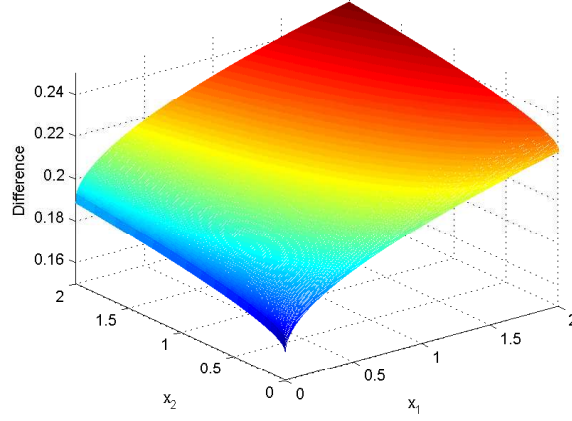


FIGURE 3.7: Difference of optimal joint utility function and the sum of utility functions in competition under $\alpha_1 = 1.3$, $\alpha_2 = 1$, $\beta_1 = 0.6$, $\beta_2 = 0.55$, $\gamma_1 = 0.9$, $\gamma_2 = 0.7$, $\sigma_1 = 0.15$, $\sigma_2 = 0.55$, $r = 0.2$, $p_1 = 1.85$, $p_2 = 1.65$, $c_1 = 1.45$, $c_2 = 1.3$ and $N = 2$

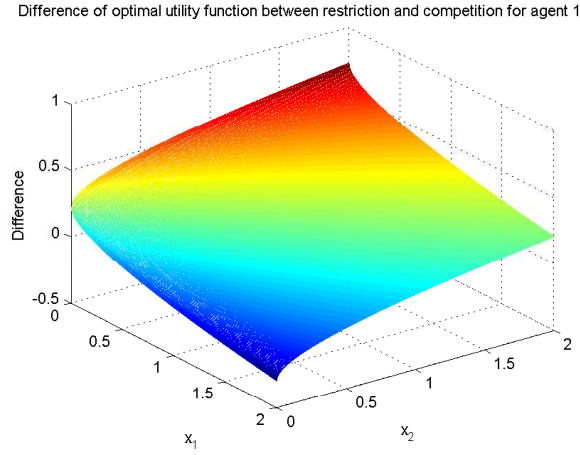


FIGURE 3.8: Difference of optimal utility functions between restriction and competition for agent 1 under $\alpha_1 = 1.3$, $\alpha_2 = 1$, $\beta_1 = 0.6$, $\beta_2 = 0.55$, $\gamma_1 = 0.9$, $\gamma_2 = 0.7$, $\sigma_1 = 0.15$, $\sigma_2 = 0.55$, $r = 0.2$, $p_1 = 1.85$, $p_2 = 1.65$, $c_1 = 1.45$, $c_2 = 1.3$ and $N = 2$

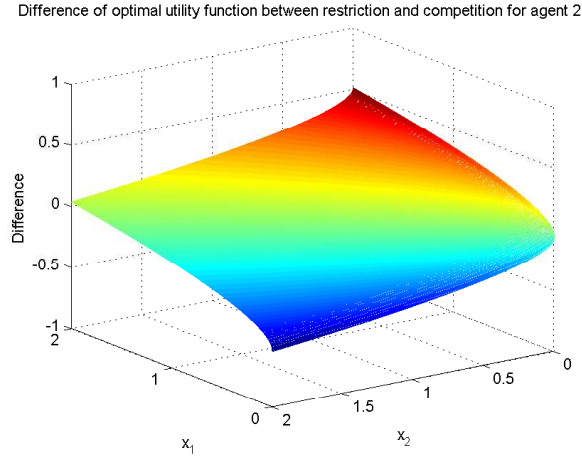


FIGURE 3.9: Difference of optimal utility functions between restriction and competition for agent 2 under $\alpha_1 = 1.3$, $\alpha_2 = 1$, $\beta_1 = 0.6$, $\beta_2 = 0.55$, $\gamma_1 = 0.9$, $\gamma_2 = 0.7$, $\sigma_1 = 0.15$, $\sigma_2 = 0.55$, $r = 0.2$, $p_1 = 1.85$, $p_2 = 1.65$, $c_1 = 1.45$, $c_2 = 1.3$ and $N = 2$

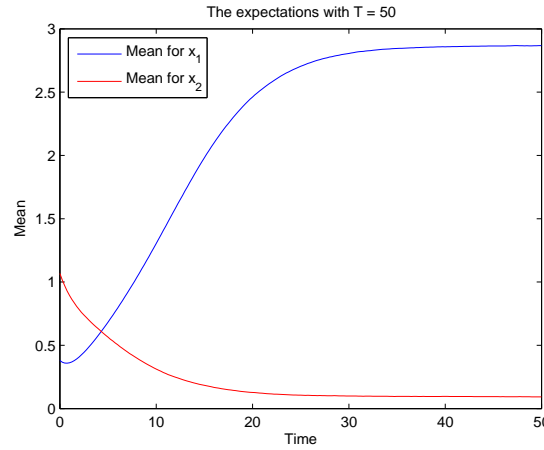


FIGURE 3.10: Means of corresponding states for competition under $\alpha_1 = 1.3$, $\alpha_2 = 1$, $\beta_1 = 0.6$, $\beta_2 = 0.55$, $\gamma_1 = 0.9$, $\gamma_2 = 0.7$, $\sigma_1 = 0.15$, $\sigma_2 = 0.55$, $r = 0.2$, $p_1 = 1.85$, $p_2 = 1.65$, $c_1 = 1.45$, $c_2 = 1.3$, $N = 2$ and $T = 50$

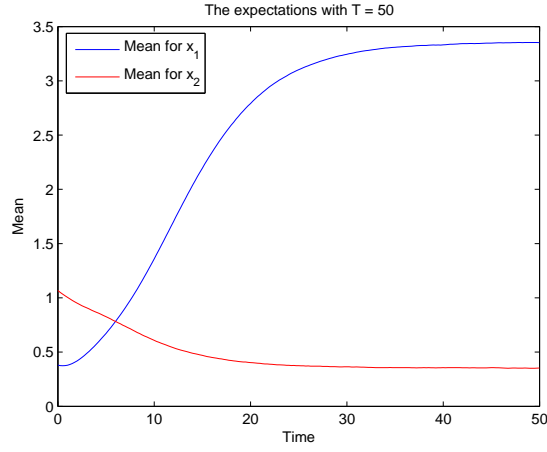


FIGURE 3.11: Means of corresponding states with restriction under $\alpha_1 = 1.3$, $\alpha_2 = 1$, $\beta_1 = 0.6$, $\beta_2 = 0.55$, $\gamma_1 = 0.9$, $\gamma_2 = 0.7$, $\sigma_1 = 0.15$, $\sigma_2 = 0.55$, $r = 0.2$, $p_1 = 1.85$, $p_2 = 1.65$, $c_1 = 1.45$, $c_2 = 1.3$ and $T = 50$

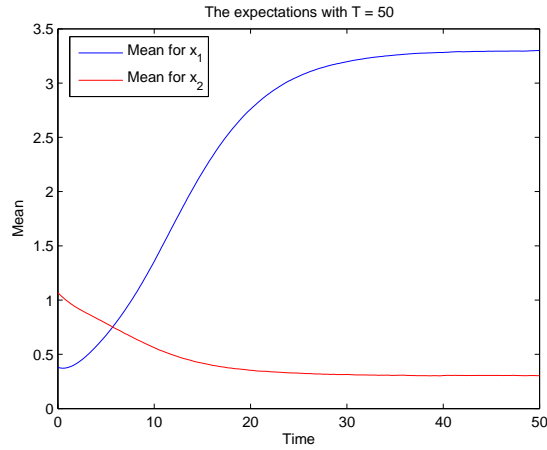


FIGURE 3.12: Means of corresponding states for the case of optimal joint utility function under $\alpha_1 = 1.3$, $\alpha_2 = 1$, $\beta_1 = 0.6$, $\beta_2 = 0.55$, $\gamma_1 = 0.9$, $\gamma_2 = 0.7$, $\sigma_1 = 0.15$, $\sigma_2 = 0.55$, $r = 0.2$, $p_1 = 1.85$, $p_2 = 1.65$, $c_1 = 1.45$, $c_2 = 1.3$ and $T = 50$

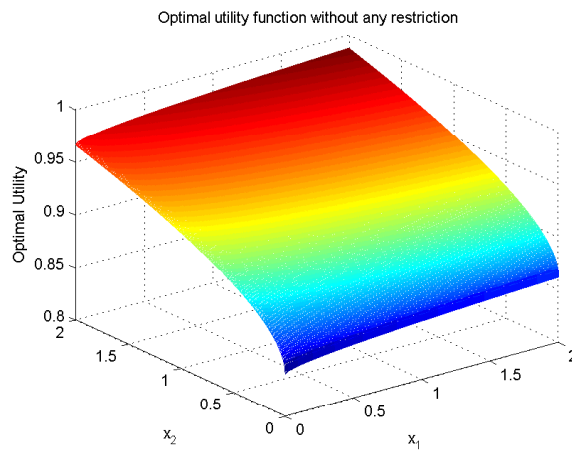


FIGURE 3.13: Optimal utility function in competition under $\alpha_1 = -1.3$, $\alpha_2 = 4$, $\gamma_1 = -0.9$, $\gamma_2 = 0.7$, $\sigma_1 = 0.15$, $\sigma_2 = 0.55$, $r = 0.2$, $p_1 = 1.85$, $p_2 = 3$, $c_1 = 1.45$, $c_2 = 2$ and $N = 2$

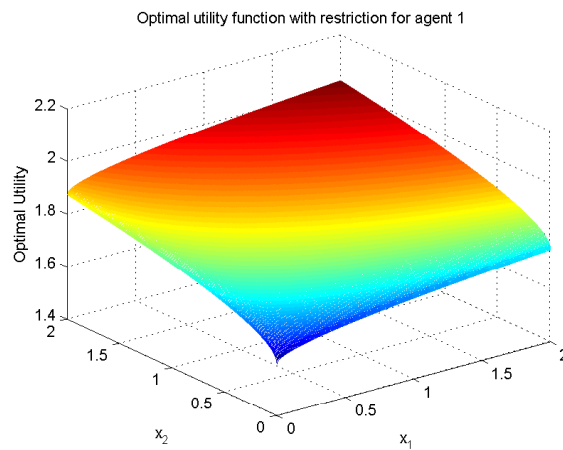


FIGURE 3.14: Optimal utility function with restriction for agent 1 under $\alpha_1 = -1.3$, $\alpha_2 = 4$, $\gamma_1 = -0.9$, $\gamma_2 = 0.7$, $\sigma_1 = 0.15$, $\sigma_2 = 0.55$, $r = 0.2$, $p_1 = 1.85$, $p_2 = 3$, $c_1 = 1.45$, $c_2 = 2$ and $N = 2$

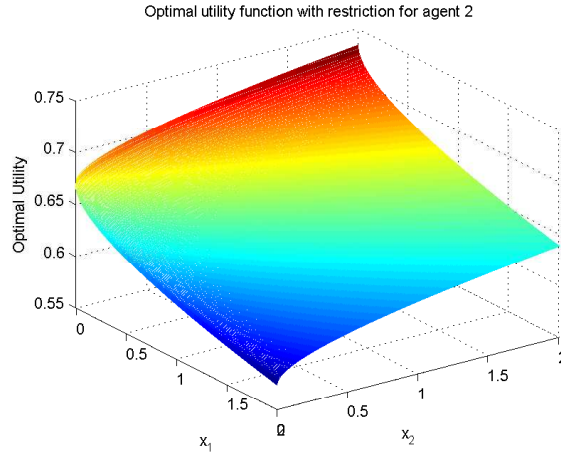


FIGURE 3.15: Optimal utility function with restriction for agent 2 under $\alpha_1 = -1.3$, $\alpha_2 = 4$, $\gamma_1 = -0.9$, $\gamma_2 = 0.7$, $\sigma_1 = 0.15$, $\sigma_2 = 0.55$, $r = 0.2$, $p_1 = 1.85$, $p_2 = 3$, $c_1 = 1.45$ and $c_2 = 2$

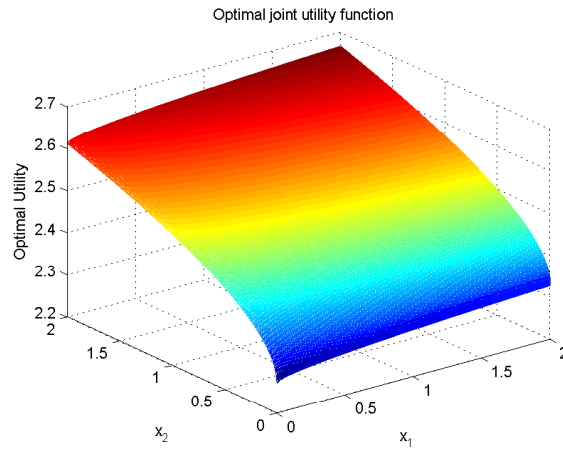


FIGURE 3.16: Optimal joint utility function under $\alpha_1 = -1.3$, $\alpha_2 = 4$, $\gamma_1 = -0.9$, $\gamma_2 = 0.7$, $\sigma_1 = 0.15$, $\sigma_2 = 0.55$, $r = 0.2$, $p_1 = 1.85$, $p_2 = 3$, $c_1 = 1.45$ and $c_2 = 2$

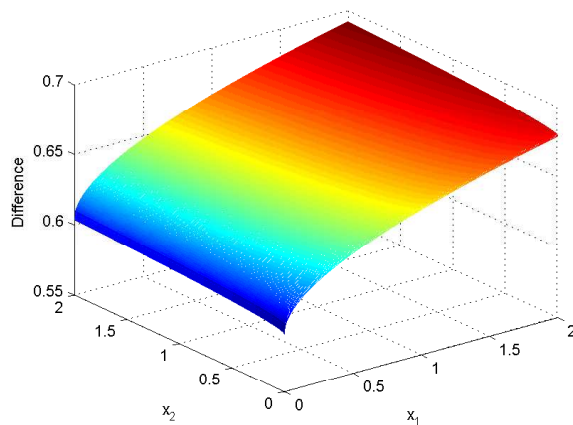


FIGURE 3.17: Difference of optimal joint utility function and the sum of utility functions in competition under $\alpha_1 = -1.3$, $\alpha_2 = 4$, $\gamma_1 = -0.9$, $\gamma_2 = 0.7$, $\sigma_1 = 0.15$, $\sigma_2 = 0.55$, $r = 0.2$, $p_1 = 1.85$, $p_2 = 3$, $c_1 = 1.45$, $c_2 = 2$ and $N = 2$

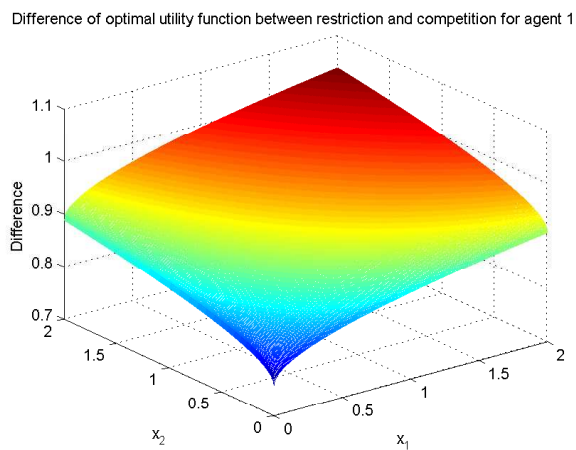


FIGURE 3.18: Difference of optimal utility functions between restriction and competition for agent 1 under $\alpha_1 = -1.3$, $\alpha_2 = 4$, $\gamma_1 = -0.9$, $\gamma_2 = 0.7$, $\sigma_1 = 0.15$, $\sigma_2 = 0.55$, $r = 0.2$, $p_1 = 1.85$, $p_2 = 3$, $c_1 = 1.45$, $c_2 = 2$ and $N = 2$

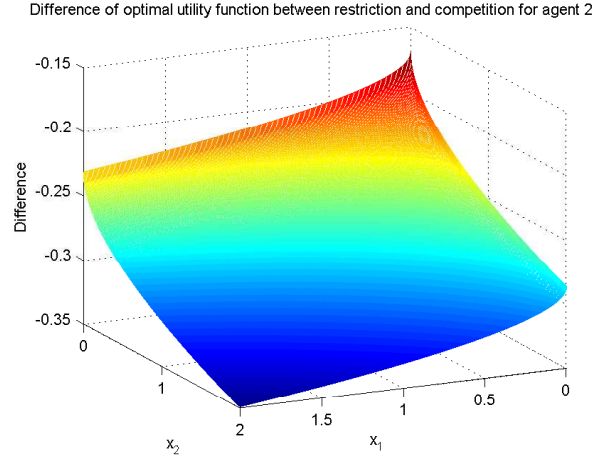


FIGURE 3.19: Difference of optimal utility functions between restriction and competition for agent 2 under $\alpha_1 = -1.3$, $\alpha_2 = 4$, $\gamma_1 = -0.9$, $\gamma_2 = 0.7$, $\sigma_1 = 0.15$, $\sigma_2 = 0.55$, $r = 0.2$, $p_1 = 1.85$, $p_2 = 3$, $c_1 = 1.45$, $c_2 = 2$ and $N = 2$

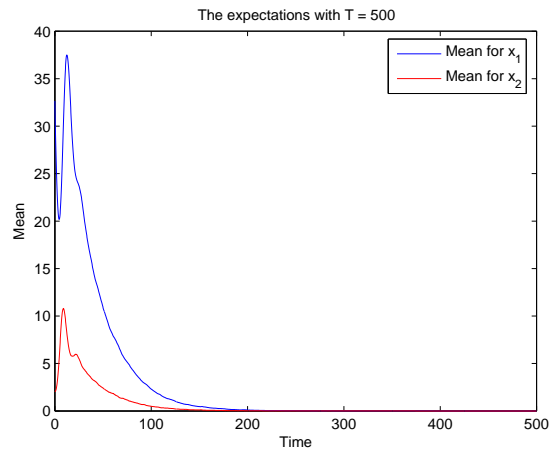


FIGURE 3.20: Means of corresponding states for competition under $\alpha_1 = -1.3$, $\alpha_2 = 4$, $\gamma_1 = -0.9$, $\gamma_2 = 0.7$, $\sigma_1 = 0.15$, $\sigma_2 = 0.55$, $r = 0.2$, $p_1 = 1.85$, $p_2 = 3$, $c_1 = 1.45$, $c_2 = 2$, $N = 2$ and $T = 500$

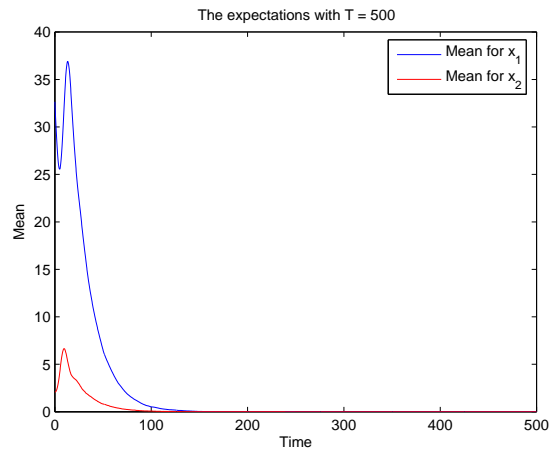


FIGURE 3.21: Means of corresponding states with restriction under $\alpha_1 = -1.3$, $\alpha_2 = 4$, $\gamma_1 = -0.9$, $\gamma_2 = 0.7$, $\sigma_1 = 0.15$, $\sigma_2 = 0.55$, $r = 0.2$, $p_1 = 1.85$, $p_2 = 3$, $c_1 = 1.45$, $c_2 = 2$, $N = 2$ and $T = 500$

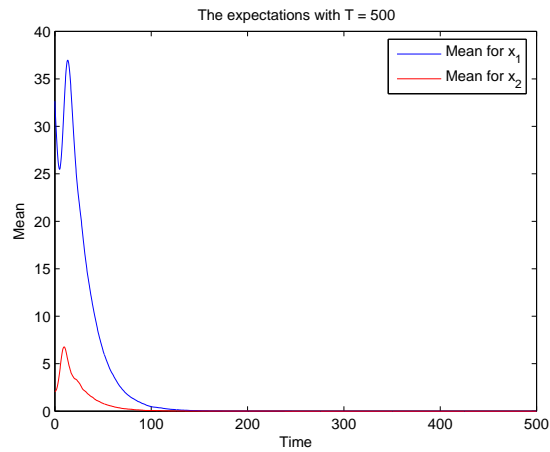


FIGURE 3.22: Means of corresponding states for the case of optimal joint utility function under $\alpha_1 = -1.3$, $\alpha_2 = 4$, $\gamma_1 = -0.9$, $\gamma_2 = 0.7$, $\sigma_1 = 0.15$, $\sigma_2 = 0.55$, $r = 0.2$, $p_1 = 1.85$, $p_2 = 3$, $c_1 = 1.45$, $c_2 = 2$, $N = 2$ and $T = 500$

Chapter 4

Sustainable Yields in Fisheries: Uncertainty, Risk-Aversion and Mean-Variance Analysis

Maximum sustainable yield models were among the first mathematical models applied to fishery economics and are now well established, see for example Clark (2006) [?]. While models such as the Schaefer model (1957) [?], which takes economic considerations such as profit taking into account, are seen as more realistic than maximum sustainable yield models, the latter are still used as a benchmark, in particular when it comes to policy implications. Evidence of this can be found in articles by Maundner (2002) [?], Jacobson et al. (2002) [?] and Roughgarden and Smith (1996) [?]. Sustainability as a concept, of course, has had somewhat of a renaissance in recent years, as people rethink their approaches to the environment, renewable resources and wildlife conservation. With the exception of Bousquet et al. (2008) [?], models taking maximal (optimal) sustainable yield as their primary objective have only been considered in a deterministic framework. The analysis in

this case is very simple. The underlying deterministic logistic growth dynamic with constant harvesting effort u is given by

$$x'(t) = \kappa x(t) (\theta - x(t)) - qux(t). \quad (4.1)$$

Without adopting any fishing effort, (4.1) has two fixed points, i.e., 0 and θ . In the case of 0, it represents the situation where the species becomes extinct. On the other hand, the fixed points θ represents the case of an ecological equilibrium. The non-zero fixed point can be easily computed, in fact

$$x(\infty, u) = \theta - \frac{qu}{\kappa}, \quad (4.2)$$

and the effort level u that maximizes this fixed point is called the maximum sustainable yield effort, here

$$u^* = \frac{\theta\kappa}{2q}. \quad (4.3)$$

The maximum sustainable yield (MSY) is then given as $MSY = qu^*x(\infty, u^*)$.

Now, it is clearly the case that fish populations do not grow deterministically, but are affected by random sources which can be caused either environmentally, e.g., climate change, or ecologically, e.g., availability of food sources or existence of predators. The issue that fish populations are affected by uncertainty was taken up in the previous Chapter. In this Chapter, the classical deterministic model has been extended by adding a level dependent diffusion term to equation (4.1). The dynamics are then governed by a stochastic differential equation and fixed points no longer exist, therefore the classical notion of maximum sustainable yield does not make sense in this context. It was shown, however, that the general concept of sustainability and maximization can be carried over to this more realistic

setup. Firstly, while stochastic differential equations seldom admit fixed points, they often admit so-called stable or equilibrium distributions. For the case of the fishery, this would mean that, once this distribution is reached, fish numbers can still fluctuate stochastically, but the underlying distribution no longer changes over time. The equilibrium distribution of a stochastic differential equation of type (4.1) is nowadays, in principle, well understood. The non-equilibrium distribution has only been computed very recently by Yang and Ewald (2008) [?] and may play a role in future work. Fisheries may now want to maximize certain functionals that depend on this equilibrium distribution. The first such functional that comes to mind is the expected value of the equilibrium distribution, leading to the concept of maximum expected sustainable yield, and this will be discussed in section 4.3. A detailed study will be made of the effect of uncertainty on the harvesting behaviour of the fishery.

In reality, it is well known that economic agents behave in a risk averse fashion and act in such a way as to trade-off between expectation and risk. It is natural to assume that fisheries are, in general, willing to accept a lower expected yield in turn for a lower level of risk. This aspect is taken into account in section 4.4 and a linear combination of expected value and variance of the equilibrium distribution of (4.1) is used as a performance measure. The problem of maximizing expected yield is also studied under a variance constraint, as well as minimizing variance, e.g. risk, under an expected sustainable yield constraint. The author and his supervisor refer to this approach as Mean-Variance Analysis of Sustainable Fisheries, since it essentially relates to aspects studied by Markowitz (1952) [?], which it is well known led to a revolution in finance and a Nobel prize.

This chapter is related to the article by Bousquet et al. (2008) [?] who also consider sustainable yields in a stochastic dynamic environment, but consider dis-

crete time, and do not reflect on issues such as risk aversion. While the continuous time setup is considered more realistic by the author, it must be emphasized that, mathematically, it is no more difficult. In fact, the available results on the equilibrium distributions of continuous time diffusions shorten the mathematical exposition significantly and allow it to appear more elegant in places, the latter, of course, being a matter of taste. Models in continuous time with the same or similar underlying diffusion processes to this thesis have been considered by various authors. However, these authors have not considered sustainability as the primary objective of the fishery. The aspect of sustainability is, for example, considered in Pindyck (1984) [?], but only in the way that the equilibrium distribution under profit maximizing strategies is computed, which of course is conceptually different than optimization of sustainable yields. Pindyck also assumes that the firms are risk neutral. Further significant contributions along the line taken by Pindyck have been made by Lande et al. (1995) [?], Alvarez and Schlepp (1998) [?] as well as Shah and Sharma (2003) [?]. It is also worth mentioning recent work by Hartman [?] in this field. He does, however, focus on profit maximization, and sustainable yields or equilibrium distributions do not play any role in this work. The author considers that this thesis makes new conceptual contributions to the current body of work, not only with respect to sustainable yields, but also and in particular with respect to the issue of risk aversion and mean-variance analysis.

4.1. The Model

It is assumed that, without interference by the fishery, the total mass of the fish population follows the stochastic logistic growth dynamic

$$dx(t) = [\kappa x(t) (\theta - x(t))] dt + \sigma x(t) dW(t), \quad (4.4)$$

with κ , θ and σ positive constants. This dynamic is basically the classical deterministic logistic growth dynamic, extended by a level dependent diffusion term. The expression $dW(t)$ represents the increment of a Brownian motion, e.g., a continuous time random walk. Equation (4.4) is, on the other hand, a geometric mean reverting process. For a full specification of (4.1) it is necessary to apply an initial condition $x(0) = x_0$ and it is assumed that $x_0 > 0$. Interestingly the precise value of x_0 will not play a role in the following sections. Given that the fishery harvests the fish following a constant effort strategy u , the actual mass rate of fish harvested is then assumed to be $qux(t)$, where $q > 0$ denotes an efficiency parameter. The dynamic for the fish population is then given by

$$dx(t) = [\kappa x(t) (\theta - x(t)) - qux(t)] dt + \sigma x(t) dW(t), \quad (4.5)$$

which is equivalent to

$$dx(t) = \kappa x(t) \left(\left\{ \theta - \frac{qu}{\kappa} \right\} - x(t) \right) dt + \sigma x(t) dW(t). \quad (4.6)$$

This has the same dynamic structure as (4.4), with the mean reverting parameter θ replaced by $\theta - \frac{qu}{\kappa}$. Note that the fishing effort influences the mean reversion parameter, and hence also the long term expectation and variance of this process, which will be discussed in the following section. Be warned at this point though, that the long term expectation does not coincide with the mean reversion parameter, Merton (1975) [?] identified this feature and called it expectation bias. It will be indicated in Chapter 5 that geometric mean reversion appears in various economic and biological models. For example, in a stochastic Solow model, the process appears as an interest rate process in the work of Merton.

4.2. Sustainability and Equilibrium distribution

The stochastic differential equation (4.6) does not have any fixed point other than 0 and the deterministic equilibrium analysis does not apply. Clearly at each point in time, $x(t)$ is random, and convergence and asymptotic behaviour then needs to be understood in terms of probabilities and distribution. Under regularity conditions which are outlined in Malliaris and Brock (1982) [?], page 106, the process $x(t)$ indeed converges in distribution to a random variable and one formally writes

$$\lim_{t \rightarrow \infty} x(t) =: x(\infty). \quad (4.7)$$

The distribution of $x(\infty)$ is then called the equilibrium distribution of (4.6). It is, in principle, possible to conclude from Merton's work, that under the condition $2\kappa\theta > \sigma^2 + 2qu$ the equilibrium distribution exists; it is independent of the starting value and essentially represents a Gamma distribution. This requires some relabeling of coefficients and transformations, as Merton's model is set up as a macro economic growth model. Alternatively, a simple derivation of the equilibrium distribution can be found in Ewald and Yang (2007) [?], equation (25). The author and his supervisor conclude the following:

Proposition 4.2.1. *Under the assumption $2\kappa\theta > \sigma^2 + 2qu$ the fish population reaches equilibrium distribution. This density function is given by*

$$\rho(x) = \begin{cases} \frac{b^\gamma}{\Gamma(\gamma)} x^{\gamma-1} e^{-bx} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

with coefficients $\gamma = \frac{2(\kappa\theta - qu)}{\sigma^2} - 1$ and $b = \frac{2\kappa}{\sigma^2}$.

If the condition $2\kappa\theta > \sigma^2 + 2qu$ is not satisfied, there is a positive probability that the population is driven to extinction. Clearly, harvesting efforts which are likely to cause extinction are considered to be non-sustainable. In terms of the harvesting effort, this gives the following sustainability condition

$$0 \leq u < \frac{2\kappa\theta - \sigma^2}{2q}. \quad (4.8)$$

It is worthwhile to note, that this sustainability condition becomes more restrictive, the higher the uncertainty parameter σ is, which of course makes sense. It is also assumed that harvesting effort is non-negative. Condition (4.8) becomes meaningless unless we assume $2\kappa\theta > \sigma^2$, a standard condition which will be assumed from now on. This assumption appears repeatedly in other mean reverting models; see, for example, Alos and Ewald (2008) [?]. In reality it may of course be possible that the condition $2\kappa\theta > \sigma^2$ is not satisfied. Reasons for this may be extremely high uncertainty, caused by factors such as global warming, pollution etc.. or low mean reversion speeds, e.g. low values of κ , which can be found in populations which reproduce at a very slow rate, for example whales or sharks. If this is the case, there is a positive probability that the population will die out, even if a zero harvesting policy is adopted.

4.3. Maximum expected sustainable yield

If the fishing effort u satisfies the sustainability condition (4.8), the corresponding expected sustainable yield (ESY) from this effort is defined as

$$ESY(u) = qu\mathbb{E}(x(\infty, u)) = qu \int_0^\infty x\rho(x, u)dx, \quad (4.9)$$

where u has been used as an additional argument in the notation to emphasize the fact that the equilibrium distribution of (4.6) depends on u . The objective in this section is to determine the value of u that maximises the expected sustainable yield. As u is assumed to be sustainable $2(\kappa\theta - qu) > \sigma^2$ and it is concluded from Ewald and Yang (2007) [?], equation (31), that the first moment of the equilibrium distribution of (4.6) is given by

$$\mathbb{E}\{x(\infty)\} = \frac{\kappa\theta - qu}{\kappa} - \frac{\sigma^2}{2\kappa}. \quad (4.10)$$

The maximum expected sustainable yield (MESY) and the maximizing fishing effort u^* can then be derived by computing the maximizer of

$$ESY(u) = qu\mathbb{E}\{x(\infty, u)\} = \frac{\kappa\theta qu - q^2 u^2}{\kappa} - \frac{\sigma^2 qu}{2\kappa}. \quad (4.11)$$

The latter is very easy as (4.11) merely presents a quadratic equation in u . The optimal fishing effort is given by

$$u^* = \frac{2\kappa\theta - \sigma^2}{4q}, \quad (4.12)$$

and the expected maximum sustainable yield is obtained by substitution as

$$\begin{aligned} MESY &= \left(\theta q - \frac{\sigma^2 q}{2\kappa}\right) \frac{2\kappa\theta - \sigma^2}{4q} - \frac{(2\kappa\theta - \sigma^2)^2}{16\kappa} \\ &= \frac{(2\kappa\theta - \sigma^2)^2}{16\kappa}. \end{aligned} \quad (4.13)$$

Note that (4.12) indeed satisfies the sustainability condition (4.8). Also note that for $\sigma = 0$ this expression coincides with expression (4.3). This of course is expected, as the case $\sigma = 0$ corresponds to the deterministic setup reviewed in the introduction

of this Chapter. However, expression (4.12) was derived using the equilibrium distribution, and the fact that in the limit case $\sigma = 0$ the deterministic expression can be derived: this implies a certain regularity property of the equilibrium distribution. It is not difficult to see, that MESY is decreasing in σ . Indeed we have

$$\frac{\partial}{\partial \sigma} q u^* E \{x(\infty)\} = -\frac{\sigma(2\kappa\theta - \sigma^2)}{4\kappa} < 0$$

for $\sigma^2 < 2\kappa\theta$, which is implied by the sustainability condition. Fisheries or fishery agencies whose objective it is to guarantee sustainability need to take this very carefully into account. On the other hand, it can easily be shown by differentiating equations (4.12) and (4.13) with respect to κ , that u^* and MESY are both increasing in κ . As κ effectively represents the speed at which the ecological system reacts, this means that extra care needs to be taken, when the target species has a low κ value, for example with whales and sharks that reproduce very slowly.

4.4. Optimal sustainable yield under risk aversion

The pure notion of sustainability, of course, already incorporates a component of risk aversion. The worst case scenario for a particular fishery is that the fish population dies out, and economic rents from the species harvested are extinguished. The fishery may, of course, move to another species, but from an ecological and bio-conservation point, this case should be avoided at all costs. The numerical example in Chapter 3 shows that, once the prey becomes extinct, the predator will also become extinct in the future. Another point, however, is that under a sustainable fishing effort, the fishery may be willing to trade-off expected sustainable yield for more certainty, e.g., less variance of the equilibrium distribution. There are several conceptually different approaches about how to incorporate risk aversion using the

variance as a measure of risk. One possibility is to use the variance as a penalty function. Alternatively, one may think of maximizing the expected sustainable yield under a variance constraint or minimizing variance of the equilibrium distribution under an expectation constraint. This thesis will examine the first point in this section and the second point in the next section. More sophisticated approaches using utility theory or more general risk measures are also possible, but at present these three elementary ways to model risk aversion will be studied. Considering the problem

$$\max_u E \{qux(\infty, u)\} - \alpha Var(qux(\infty, u)), \quad (4.14)$$

where $\alpha \geq 0$ represents the level of risk aversion. The problem will be solved by taking advantage of the analytic formulas for first and second moments of the equilibrium distribution of geometric mean reversion in Ewald and Yang (2008) [?]. It is concluded that

$$\begin{aligned} & E \{qux(\infty, u)\} - \alpha Var(qux(\infty, u)) \\ &= qu\mathbb{E}\{x(\infty, u)\} - \alpha q^2 u^2 \mathbb{E}\{x^2(\infty, u)\} + \alpha q^2 u^2 (\mathbb{E}\{x(\infty, u)\})^2 \\ &= \frac{\alpha \sigma^2 q^3}{2\kappa^2} u^3 + \left(\frac{\alpha \sigma^4 q^2}{4\kappa^2} - \frac{\alpha \theta \sigma^2 q^2}{2\kappa} - \frac{q^2}{\kappa} \right) u^2 + \left(\theta q - \frac{\sigma^2 q}{2\kappa} \right) u. \end{aligned} \quad (4.15)$$

Differentiating the latter equation with respect to u and setting the derivative equal to 0, gives as necessary condition for the optimal fishing effort

$$\frac{3\alpha \sigma^2 q^3}{2\kappa^2} u^2 - \left(\frac{\alpha \sigma^2 q^2 A}{\kappa} + \frac{2q^2}{\kappa} \right) u + qA = 0, \quad (4.16)$$

where $A = \theta - \frac{\sigma^2}{2\kappa} > 0$. Equation (4.16) has two positive roots since

$$\left(\frac{\alpha \sigma^2 q^2 A}{\kappa} + \frac{2q^2}{\kappa} \right)^2 - \frac{6\alpha \sigma^2 q^4 A}{\kappa^2} = \frac{q^4 (\alpha \sigma^2 A - 1)^2}{\kappa^2} + \frac{3q^4}{\kappa^2} \geq 0.$$

These roots are given by

$$u_{\pm} = \frac{\kappa}{3\alpha\sigma^2q} \left[(\alpha\sigma^2A + 2) \pm \sqrt{(\alpha\sigma^2A - 1)^2 + 3} \right].$$

To find out which positive root is the maximizer, the second derivative of equation (4.15) is computed with respect to u as

$$\frac{3\alpha\sigma^2q^3}{\kappa^2}u - \left(\frac{\alpha\sigma^2q^2A}{\kappa} + \frac{2q^2}{\kappa} \right).$$

It can be seen that only u_- causes the second derivative of equation (4.15) to be negative, which therefore implies that u_- is the maximizer. Applying de l'Hospital's rule, it can easily be seen that in the limit for α tending to 0, this gives expression (4.13) from u_- . This of course is as it should be, as the case $\alpha = 0$ is the case where there is no explicit risk aversion. Furthermore it can be shown that u_- is decreasing in α and increasing in κ . Indeed,

$$\frac{\partial}{\partial \alpha} u_- = \frac{\kappa \left(-2\sqrt{(\alpha\sigma^2A - 1)^2 + 3} + 4 - \alpha\sigma^2A \right)}{3\alpha^2\sigma^2q\sqrt{(\alpha\sigma^2A - 1)^2 + 3}} \leq 0.$$

As (4.13) is sustainable and with α increasing and the fishing effort decreasing, the following is obtained

$$u^* = u_- = \frac{\kappa}{3\alpha\sigma^2q} \left[(\alpha\sigma^2A + 2) - \sqrt{(\alpha\sigma^2A - 1)^2 + 3} \right] \quad (4.17)$$

is indeed sustainable for all $\alpha > 0$ and therefore represents the optimal sustainable fishing effort. Substituting u^* into equation (4.15), gives the optimal expected sustainable yield (OESY). It can be seen that in the range $2\kappa\theta > \sigma^2 \geq \kappa\theta$ the

optimal fishing effort u^* is decreasing in σ . Indeed,

$$\frac{\partial}{\partial \sigma} u^* = \frac{\kappa}{3\alpha\sigma^3 q} \left\{ \left[\sigma \left(2\alpha\sigma A + \alpha\sigma^2 \frac{\partial}{\partial \sigma} A \right) \left(1 - \frac{\alpha\sigma^2 A - 1}{\sqrt{(\alpha\sigma^2 A - 1)^2 + 3}} \right) \right] - 2 \left[(\alpha\sigma^2 A + 2) - \sqrt{(\alpha\sigma^2 A - 1)^2 + 3} \right] \right\}$$

and $2\alpha\sigma A + \alpha\sigma^2 \frac{\partial}{\partial \sigma} A \leq 0$ if $\sigma^2 \in [\kappa\theta, 2\kappa\theta)$. As seen above, the optimal fishing effort is decreasing in the level of risk aversion. While the mathematics behind this result is sound, its intuition is not trivial. It cannot a priori be said that more risk aversion causes lower optimal fishing effort. There are essentially two effects here, a higher fishing effort potentially leads to a lower level of the population, and hence a lower variance; however, on the other hand, yields become higher and the variance of the yield may, in fact, increase. These effects are also traded off, with a similar effect applying to expectation. The result obtained here nevertheless clearly shows that more risk aversion causes lower optimal fishing effort. The analysis carried out in this section can easily be extended to take higher moments of the equilibrium distribution into account. As indicated earlier, all moments can easily be computed from Merton (1975) [?] or iteratively from Ewald and Yang (2007) [?], and these moments can be used to construct more complex risk measures. For example it is possible to study exponential utility and in fact as non-integer moments are also available, the case of constant relative risk aversion. These cases will be considered in future work.

4.5. Mean-Variance Analysis of Sustainable Yields

This section will consider an approach which accounts for risk in a slightly different way than in the previous section. In finance, this approach is classically

known as mean-variance analysis and thus it will be called the mean variance analysis of sustainable yields. It is believed that this approach has never been applied to sustainable yields fishery models before. There are, in principle, two related problems. These are:

- maximization of expected sustainable yield under limited risk, e.g., variance.
- minimization of risk under guaranteed minimum level of expected sustainable yield.

Considering the first problem, e.g.

$$\max_u E \{qux(\infty)\},$$

subject to

$$\begin{aligned} Var(qux(\infty)) &\leq L, \\ 2\kappa\theta - 2qu - \sigma^2 &\geq 0, \end{aligned}$$

where $L > 0$ is a constant representing the maximum acceptable risk and the second constraint is the sustainability condition (4.8); the latter system is equivalent to

$$\min_u \frac{q^2}{\kappa} u^2 - \left(\theta - \frac{\sigma^2}{2\kappa} \right) qu$$

subject to

$$\begin{aligned} \frac{\sigma^2 q^3}{2\kappa^2} u^3 - \left(\frac{\theta\sigma^2}{2\kappa} - \frac{\sigma^4}{4\kappa^2} \right) q^2 u^2 + L &\geq 0, \\ -2qu + 2\kappa\theta - \sigma^2 &\geq 0. \end{aligned}$$

The Kuhn-Tucker theorem will be used to solve this constraint optimization problem. The Kuhn-Tucker conditions are

$$\begin{aligned} \frac{2q^2}{\kappa}u - \left(\theta - \frac{\sigma^2}{2\kappa}\right)q - \left[\frac{3\sigma^2q^3}{2\kappa^2}u^2 - 2\left(\frac{\theta\sigma^2}{2\kappa} - \frac{\sigma^4}{4\kappa^2}\right)q^2u\right]\mu_1 + 2q\mu_2 &= 0 \\ \left[\frac{\sigma^2q^3}{2\kappa^2}u^3 - \left(\frac{\theta\sigma^2}{2\kappa} - \frac{\sigma^4}{4\kappa^2}\right)q^2u^2 + L\right]\mu_1 &= 0 \\ (-2qu + 2\kappa\theta - \sigma^2)\mu_2 &= 0 \end{aligned} \quad (4.18)$$

where μ_1 and μ_2 are non negative. There are four cases to consider:

1. $\mu_1 = \mu_2 = 0 \Rightarrow u^* = \frac{\kappa}{2q} \left(\theta - \frac{\sigma^2}{2\kappa}\right)$
2. $\mu_1 \neq 0, \mu_2 = 0 \Rightarrow u^*$ is a positive solution for the cubic function defined by constraint 1
3. $\mu_1 = 0, \mu_2 \neq 0 \Rightarrow u^* = \frac{2\kappa\theta - \sigma^2}{2q}$
4. $\mu_1 \neq 0, \mu_2 \neq 0 \Rightarrow u^* = \frac{2\kappa\theta - \sigma^2}{2q}$, for some L .

Cases 3 and 4 do not provide sustainable yields, but in order to apply the Kuhn-Tucker theorem, it is necessary to formally allow for these cases. Nevertheless, they would give $E\{qu^*x(\infty)\} = 0 = Var(qu^*x(\infty))$ which excludes both cases from providing maximizers. Note that in the range of sustainable yields $0 \leq u < \frac{2\kappa\theta - \sigma^2}{2q}$ the variance term $Var(qux(\infty))$ is bounded and therefore, if L is chosen to be sufficient large, the variance constraint becomes meaningless. In fact this is the case when $L \geq \frac{4\kappa\sigma^2}{27} \left(\theta - \frac{\sigma^2}{2\kappa}\right)^3$. Then case 1 applies and $\frac{\kappa}{2q} \left(\theta - \frac{\sigma^2}{2\kappa}\right)$ is the maximizer. Note that this level of fishing effort coincides with (4.12). This is no surprise, because the acceptable level of risk L is higher than that which can actually be caused by the fish population under any fishing effort and agents effectively become risk neutral. If however the acceptable level of risk L is lower than $\frac{4\kappa\sigma^2}{27} \left(\theta - \frac{\sigma^2}{2\kappa}\right)^3$,

then the maximizer is defined via case 2, and it is necessary to solve

$$\frac{\sigma^2 q^3}{2\kappa^2} u^3 - \left(\frac{\theta \sigma^2}{2\kappa} - \frac{\sigma^4}{4\kappa^2} \right) q^2 u^2 + L = 0 \quad (4.19)$$

and obtain μ_1 from the first equation in (5.18). Then, since $\frac{\sigma^2 q^3}{2\kappa^2} u^3 - \left(\frac{\theta \sigma^2}{2\kappa} - \frac{\sigma^4}{4\kappa^2} \right) q^2 u^2$ is decreasing when $u \in \left[0, \frac{2\kappa}{3q} \left(\theta - \frac{\sigma^2}{2\kappa} \right) \right]$, the equation will lead to a maximizer which is less than $\frac{\kappa}{2q} \left(\theta - \frac{\sigma^2}{2\kappa} \right)$. Note that the cubic (4.19) always has a unique positive root if $L < \frac{4\kappa\sigma^2}{27} \left(\theta - \frac{\sigma^2}{2\kappa} \right)^3$. The reason for this is that the function $\frac{\sigma^2 q^3}{2\kappa^2} u^3 - \left(\frac{\theta \sigma^2}{2\kappa} - \frac{\sigma^4}{4\kappa^2} \right) q^2 u^2$ is decreasing, continuous and has a sign-change on the interval $\left[0, \frac{\kappa}{2q} \left(\theta - \frac{\sigma^2}{2\kappa} \right) \right]$. As there are analytic formulas for the solution of cubic equations, (4.19) can in principle be solved. The explicit expression is omitted at this point, though, because it is quite lengthy and, due to its complexity, it is difficult to analyse. If the maximizer is given via case 1, it has already been proved in section 4.3 that both u^* and MESY are increasing in κ . If the maximizer is given via case 2, e.g. lower risk tolerance L , and $\sigma^2 \in [\kappa\theta, 2\kappa\theta)$, then since $\frac{\sigma^2 q^3}{2\kappa^2}$ and $\left(\frac{\theta \sigma^2}{2\kappa} - \frac{\sigma^4}{4\kappa^2} \right) q^2$ are decreasing and increasing, respectively, u^* is decreasing. Now, considering the second, dual approach, where risk, e.g. variance, is minimized when keeping expected sustainable yield above a certain level. More precisely, it is assumed that the fishery tries to solve the following constraint optimization problem:

$$\min_u \text{Var}(qux(\infty)),$$

subject to

$$\begin{aligned} E\{qux(\infty)\} &\geq L, \\ 2\kappa\theta - 2qu - \sigma^2 &\geq 0, \end{aligned}$$

where $L \geq 0$ is the lowest expectation that is agreeable for the fishery. This system is equivalent to

$$\min_u -\frac{\sigma^2 q^3}{2\kappa^2} u^3 + \left(\frac{\theta \sigma^2}{2\kappa} - \frac{\sigma^4}{4\kappa^2} \right) q^2 u^2,$$

subject to

$$\begin{aligned} -\frac{q^2}{\kappa} u^2 + \left(\theta - \frac{\sigma^2}{2\kappa} \right) qu - L &\geq 0, \\ -2qu + 2\kappa\theta - \sigma^2 &\geq 0. \end{aligned}$$

It is necessary to assume that $\left(\theta - \frac{\sigma^2}{2\kappa} \right)^2 \geq \frac{4L}{\kappa}$, as otherwise the first constraint does not allow any sustainable fishing efforts. This condition follows from the boundedness from above, of the expected sustainable yields, and if L is above that boundary the expected sustainable yields simply cannot be achieved. The Kuhn-Tucker conditions are then

$$\begin{aligned} -\frac{3\sigma^2 q^3}{2\kappa^2} u^2 + 2 \left(\frac{\theta \sigma^2}{2\kappa} - \frac{\sigma^4}{4\kappa^2} \right) q^2 u - \left[-\frac{2q^2}{\kappa} u + \left(\theta - \frac{\sigma^2}{2\kappa} \right) q \right] \mu_1 + 2q\mu_2 &= 0 \\ \left[-\frac{q^2}{\kappa} u^2 + \left(\theta - \frac{\sigma^2}{2\kappa} \right) qu - L \right] \mu_1 &= 0 \quad (4.20) \\ (-2qu + 2\kappa\theta - \sigma^2) \mu_2 &= 0 \end{aligned}$$

where μ_1 and μ_2 are non negative. There are again four possible cases:

1. $\mu_1 = \mu_2 = 0 \Rightarrow u^* = 0$ or $u^* = \frac{2\kappa}{3q} \left(\theta - \frac{\sigma^2}{2\kappa} \right),$
2. $\mu_1 \neq 0, \mu_2 = 0 \Rightarrow u^* = \frac{\left(\theta - \frac{\sigma^2}{2\kappa} \right) \pm \sqrt{\left(\theta - \frac{\sigma^2}{2\kappa} \right)^2 - \frac{4L}{\kappa}}}{\frac{2q}{\kappa}},$
3. $\mu_1 = 0, \mu_2 \neq 0 \Rightarrow u^* = \frac{2\kappa\theta - \sigma^2}{2q},$
4. $\mu_1 \neq 0, \mu_2 \neq 0 \Rightarrow u^* = \frac{2\kappa\theta - \sigma^2}{2q},$ for some L .

As before, cases 3 and 4 imply $\mathbb{E}\{qu^*x(\infty)\} = \text{Var}(qu^*x(\infty))=0$, which is not an option, unless $L = 0$, which is unrealistic. It can be proved that the objective functional is increasing when $u \in \left[0, \frac{2\kappa}{3q} \left(\theta - \frac{\sigma^2}{2\kappa}\right)\right]$, which in case 1 would imply that $u^* = 0$. This however again implies $\mathbb{E}\{qu^*x(\infty)\} = 0$ which does not satisfy the constraint, unless $L = 0$. Therefore, case 2 applies in all non-trivial cases. Clearly

$$u^* \in \left[\frac{\left(\theta - \frac{\sigma^2}{2\kappa}\right) - \sqrt{\left(\theta - \frac{\sigma^2}{2\kappa}\right)^2 - \frac{4L}{\kappa}}}{\frac{2q}{\kappa}}, \frac{\left(\theta - \frac{\sigma^2}{2\kappa}\right) + \sqrt{\left(\theta - \frac{\sigma^2}{2\kappa}\right)^2 - \frac{4L}{\kappa}}}{\frac{2q}{\kappa}} \right]$$

satisfies the constraint, the second equation in (4.20). Substituting $\frac{\left(\theta - \frac{\sigma^2}{2\kappa}\right) \pm \sqrt{\left(\theta - \frac{\sigma^2}{2\kappa}\right)^2 - \frac{4L}{\kappa}}}{\frac{2q}{\kappa}}$ into the objective functional, gives

$$\frac{\sigma^2 \kappa}{16} \left[\frac{\left(\theta - \frac{\sigma^2}{2\kappa}\right) \pm \sqrt{\left(\theta - \frac{\sigma^2}{2\kappa}\right)^2 - \frac{4L}{\kappa}}}{\frac{2q}{\kappa}} \right]^2 \left[\frac{\left(\theta - \frac{\sigma^2}{2\kappa}\right) \mp \sqrt{\left(\theta - \frac{\sigma^2}{2\kappa}\right)^2 - \frac{4L}{\kappa}}}{\frac{2q}{\kappa}} \right]$$

and it can be seen that the minimizer is given by $\frac{\left(\theta - \frac{\sigma^2}{2\kappa}\right) - \sqrt{\left(\theta - \frac{\sigma^2}{2\kappa}\right)^2 - \frac{4L}{\kappa}}}{\frac{2q}{\kappa}}$. Therefore, the risk-minimizing sustainable effort with agreeable expected sustainable yield L is given by

$$u^* = \frac{\left(\theta - \frac{\sigma^2}{2\kappa}\right) - \sqrt{\left(\theta - \frac{\sigma^2}{2\kappa}\right)^2 - \frac{4L}{\kappa}}}{\frac{2q}{\kappa}}. \quad (4.21)$$

Differentiating the minimizer with respect to σ , gives

$$\frac{\partial}{\partial \sigma} u^* = \frac{\sigma}{2q} \left[1 - \frac{\theta - \frac{\sigma^2}{2\kappa}}{\sqrt{\left(\theta - \frac{\sigma^2}{2\kappa}\right)^2 - \frac{4L}{\kappa}}} \right] \leq 0,$$

and it can be observed, again, that the higher the uncertainty is, the lower the optimal effort. On the other hand, in order to study how κ affects the optimal fishing effort, it is necessary to compute $\frac{\partial}{\partial \kappa} u^*$. Indeed,

$$\begin{aligned} \frac{\partial}{\partial \kappa} u^* &= \frac{\partial}{\partial \kappa} \frac{\left(2\kappa\theta - \sigma^2 - \sqrt{(2\kappa\theta - \sigma^2)^2 - 16\kappa L}\right)}{4q} \\ &= \frac{1}{4q} \left[2\theta - \frac{4\theta(2\kappa\theta - \sigma^2) - 16\kappa L}{2\sqrt{(2\kappa\theta - \sigma^2)^2 - 16\kappa L}} \right] \\ &= \frac{1}{4q} \left[2\theta \left(1 - \frac{2\kappa\theta - \sigma^2}{\sqrt{(2\kappa\theta - \sigma^2)^2 - 16\kappa L}} \right) + \frac{8L}{\sqrt{(2\kappa\theta - \sigma^2)^2 - 16\kappa L}} \right] \geq 0. \end{aligned}$$

The physical interpretation of this is that, when the fishery objective is to minimize the variance and the mean reversion κ goes up, the population of fish recovers faster and this causes the variance to be larger. Therefore, fishery agents would prefer to harvest more fish to reduce the variance. In the authors' point of view, this last approach is the most conservative, as the objective here is really a minimization of risk, e.g. variance, which can be interpreted as keeping the population stable, under an agreeable minimum expected sustainable yield.

4.6. Numerical Illustration

To illustrate the effect of parameter choice, in particular the uncertainty parameter σ , numerical examples will be provided for each case considered in sections 4.3, 4.4, and 4.5. First assuming that $\theta = 1$ and $q = 0.6$. Figure 4.1 shows MESY as a function of κ and σ . It can be seen that MESY is increasing in κ and decreasing in σ , as we have concluded in section 4.3. Figure 4.2 shows that OESY for approach 1 has similar properties to MESY. The case where the fishery maximizes expected

yield with variance bounded by a given level L , which is sufficiently low to avoid a trivial case, is represented in Figure 4.3. It can be seen in this case that the optimal fishing effort is decreasing in σ and increasing in κ . Moreover it can be seen, that a larger σ offsets the effect of an increasing mean reversion speed κ , i.e., u^* increases in κ significantly more under a lower σ than under a high σ . For the case where the fisheries' objective is minimization of the variance under a minimum agreeable expected sustainable yield, it can be seen from Figure 4.4 that the optimal fishing effort is increasing in κ and σ .

4.7. Conclusion

This work extends the classical logistic growth-based sustainable yield model to accommodate uncertainty in terms of a level dependent uncertainty term. The notion of sustainable yield is introduced in this context, relying on the results of the equilibrium distribution of geometric mean reversion, and an expression for the maximum expected sustainable yield has been derived. Furthermore, the case of risk averse fisheries, which balance expected sustainable yield with risk, has been considered. This has been measured in terms of the variance of the equilibrium distribution and the effect of risk aversion on optimal sustainable yields. Finally, the concept of mean-variance analysis in sustainable fisheries was introduced and the optimal fishing efforts were derived in this context.

Graphical Illustration

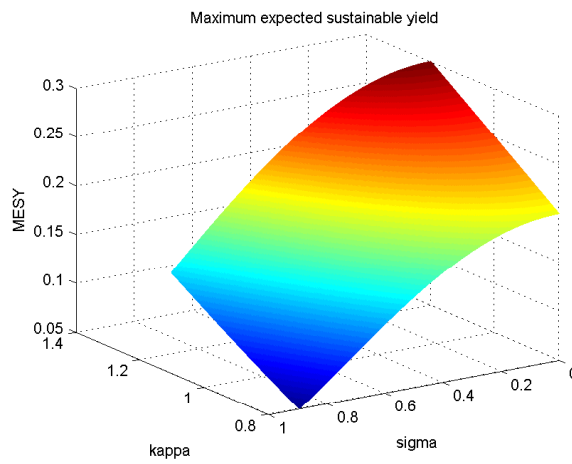


FIGURE 4.1: The maximum expected sustainable yield when $\theta = 1$ and $q = 0.6$

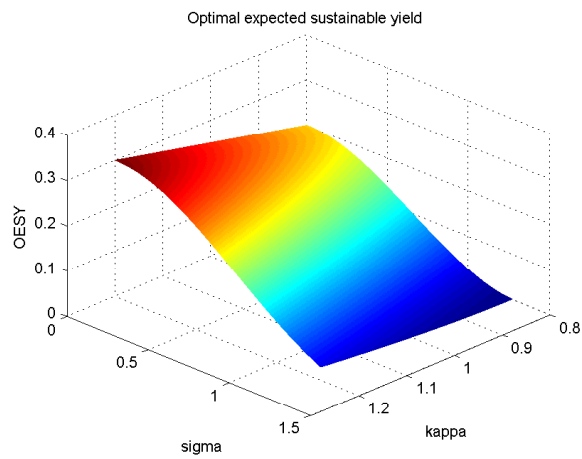


FIGURE 4.2: The optimal expected sustainable yield under risk aversion when $\alpha = 1$, $\theta = 1$ and $q = 0.6$

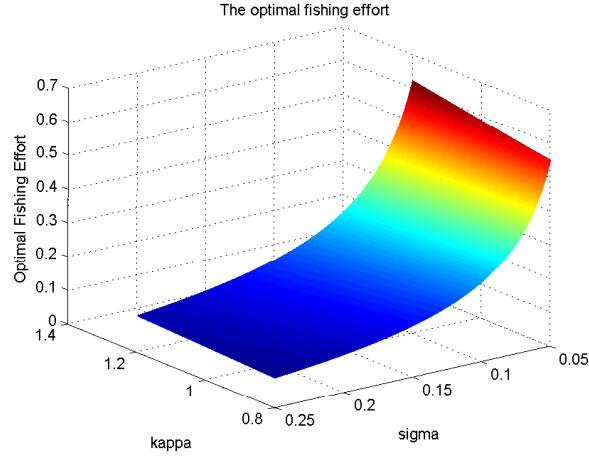


FIGURE 4.3: The optimal fishing effort for maximizing $E\{qux(\infty)\}$ under $Var(qux(\infty)) \leq L$ when $L = 10^{-4}$, $\theta = 1$ and $q = 0.6$

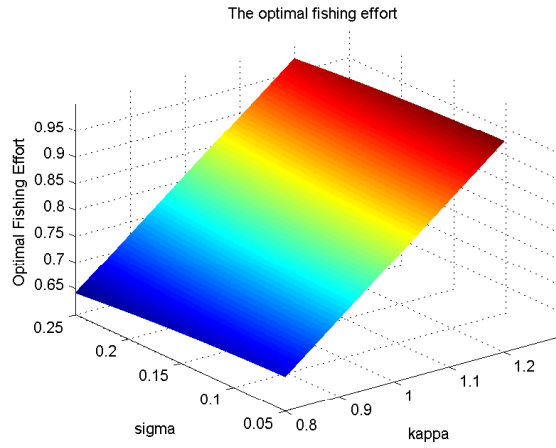


FIGURE 4.4: The optimal fishing effort for minimizing $Var(qux(\infty))$ under $E\{qux(\infty)\} \geq L$ when $L = 0.5$, $\theta = 1$ and $q = 0.6$

Chapter 5

Irreversible investment with Cox-Ingersoll-Ross type mean reversion

In this chapter, it is supposed that a financial manager faces an infinite time horizon and that the discount rate is either given exogenously or is implied by the existence of a complete spanning asset. This is in common with the approach taken in [?]. It was indicated in section 1.1 that one application of real option theory in environmental economics is to model the consumption of resources. Dixit and Pindyck (1994) presented an example where they considered that the price of oil followed a mean reverting process, see [?], Chapter 12. The main difference that will be seen in this exposition is that it is assumed that the value of the underlying financial project follows a Cox-Ingersoll-Ross process. In most of the previous work, it was assumed that the underlying financial project follows a geometric Brownian motion. It has been argued, however, by many influential authors in the economic literature that geometric Brownian motion, although convenient and mathematically easy to

handle, is not a realistic assumption for the project value process: see, for example, Dixit and Pindyck (1994) [?], Epstein et al. (1998) [?] and Metcalf and Hassett (1995) [?]. Basic microeconomic theory explains that, in the long run the price of a commodity ought to be tied to its long-run marginal cost. This feature does not exist in geometric Brownian motion models, but is present in so called mean reverting models.

Mean reverting models that have been discussed in the literature include Ornstein-Uhlenbeck, exponential Ornstein-Uhlenbeck and geometric mean reversion. The case of an Ornstein-Uhlenbeck process, i.e., a process follows

$$dx(t) = \theta (\mu - x(t)) dt + \sigma dW(t),$$

where $W(t)$ is a Wiener process, is highly unrealistic as it leads to negative values of the project value. The remedy for this is to take an exponential of an Ornstein-Uhlenbeck process, but the mean reverting structure obtained from this remains questionable. The most favoured model is the so called geometric mean reversion process, which is sometimes modified by including an economic growth factor, see Metcalf and Hassett (1995) [?]. For this process, explicit solutions for the irreversible investment problem in terms of the Kummer M function (also called confluent hypergeometric function) have been derived by Dixit and Pindyck, while empirical studies have been undertaken by Metcalf and Hassett (1995) [?]. It can be shown that the geometric mean reversion process always stays positive and, for suitable parameter constraints, has good analytical properties, such as the existence of an equilibrium distribution, see Ewald and Yang (2007) [?].

Geometric mean reversion can be interpreted in two ways: the first is that the expected relative rate of return of the project value is actually mean reverting, the second is that the mean reversion speed is proportional to the project value; however,

this setup is not economically reasonable in all cases. One process which guarantees a positive project value and has a constant mean reversion speed is the so called Cox-Ingersoll-Ross process, which will be studied in detail in section 5.1. This process which was introduced by Cox, Ingersoll and Ross (1985) in [?] has been studied in great detail in the context of interest rate models and stochastic volatility models in finance. Nevertheless, it has never been used in the context of real option models. In this chapter, the explicit solution is derived for the irreversible investment problem under the assumption that the project value follows a Cox-Ingersoll-Ross process. The solution shows various similarities to the case of the geometric mean reversion case and these are explored by comparative analysis. Certain aspects are then illustrated, which the authors believe should lead to a general preference for the Cox-Ingersoll-Ross process with regards to the geometric mean reversion process.

5.1. Some facts about the Cox-Ingersoll-Ross process

A stochastic process which follows the dynamics

$$dV(t) = \kappa (\theta - V(t)) dt + \sigma \sqrt{V(t)} dW(t), \quad (5.1)$$

with constants $\kappa, \theta, \sigma > 0$ and a Wiener process $W(t)$ is generally referred to as Cox-Ingersoll-Ross process, although other names are used from time to time. These names include, for example, Heston volatility and the square-root process. It can be shown that the process (5.1) is effectively a re-parameterized squared Bessel process (see for example Alos and Ewald (2008) [?]) and has carefully been studied by various authors since the early 1950's, most rigorously by Yor (1992) [?]. The process was introduced into finance as a model for short rate interest by Cox, Ingersoll and Ross (1985) in [?] to replace the existing Vasicek model. The Vasicek model

suffered from the fact that the interest rate can become negative, a phenomenon that has only been observed once in Japan, see [?]. Usage of the Cox-Ingersoll-Ross process as a model for the short rate leads to an affine term structure model which can easily be calibrated and this was the key to its success. On a different level, the same process was used by Heston several years later, to model the volatility of a stock, see Heston (1993) [?]. The resulting stochastic volatility model has become known as the Heston model. When compared to the Black-Scholes model, it has the advantage that it produces volatility smiles, while still being accessible and allowing analytic formulas for plain vanilla options.

A sophisticated look at the dynamics of (5.1) seems to indicate two potential problems. The square-root function is not defined for negative values and has infinite slope at 0. The first problem could potentially be cured by taking the absolute value inside the square root on the right hand side of equation (5.1). The classical existence and uniqueness results for solutions of stochastic differential equations, however, fail, as they assume a Lipschitz condition for the coefficients which is clearly violated. Using results of Yamada and Watanabe, see Karatzas (1988) [?], Chapter 5 Proposition 2.18., it can be shown, however, that in the case where the coefficient condition

$$2\kappa\theta > \sigma^2 \tag{5.2}$$

holds, a unique strong solution to (5.1) exists for arbitrary positive initial conditions $V(0) = V_0 > 0$ and that the process remains strictly positive for all times with probability one. The latter can be concluded from the fact that, under condition (5.2), the Cox-Ingersoll-Ross process is a reparameterized squared Bessel process of dimension greater than 2. The Cox-Ingersoll-Ross process has various other desirable properties such as smoothness in the Malliavin sense, see Alos and Ewald (2008) [?], as well as the existence of an equilibrium distribution which is centered

around the mean reversion level θ . In particular, examining expectation in (5.1) shows that it satisfies

$$E\{V(t)\} = V_0 + \int_0^t \kappa(\theta - E\{V(s)\}) ds,$$

which therefore implies the ordinary differential equation

$$\frac{d}{dt}E\{V(t)\} = \kappa(\theta - E\{V(t)\}), \quad (5.3)$$

with the initial value condition $E\{V(0)\} = V_0$. Equation (5.3) can be solved analytically and leads in particular to

$$E\{V(t)\} = \theta + (V_0 - \theta)e^{-\kappa t}. \quad (5.4)$$

Therefore $\lim_{t \rightarrow \infty} E\{V(t)\} = \theta$. This fact makes it possible to interpret the θ as the parameter which represents the demand/supply market clearing feature in equilibrium. As Ewald and Yang (2007) showed in [?], the popular geometric mean reversion model, which they also discuss in section 4, does not satisfy this criterion.

The equilibrium distribution of geometric mean reversion is shifted away from the mean reversion level θ and the expectation is convergent to $\theta - \frac{\sigma^2}{\kappa}$, which is extremely sensitive to changes in the volatility σ or mean reversion speed κ . As they also have shown in [?], geometric mean reversion stays strictly positive automatically without referring to any further coefficient condition, but if the value theoretically falls to zero (for example by imposing a shock), it remains there forever; if the same happens to the Cox-Ingersoll-Ross process, the infinitesimal drift $\kappa\theta dt$ will push the process back to a positive value and cause the system to recover. This effect may be economically reasonable or not. One example is that if the process

models the value of a company, a value of 0 could be interpreted as bankruptcy and all further considerations about the future of this company may end; on the other hand, another investor may take over the company and bring it back to life if the stock of the resources is still sufficient. In the case where the reserve of resources is undeveloped, it may not be realistic that the current value of the resources of zero causes the value to remain at zero in the future. Putting these hypothetical considerations aside, on a technical level, the Cox-Ingersoll-Ross process produces more continuity and regularity at $V = 0$ than the geometric mean reversion process does. Because of these technical aspects, the Cox-Ingersoll-Ross process is preferred over geometric mean reversion. Further arguments will be presented later during the quantitative analysis.

5.2. Irreversible Investment

The following problem should be considered. The value of an investment project is given by a stochastic process which follows the dynamics

$$dV(t) = \alpha(V(t))dt + \sigma(V(t))dW(t), \quad (5.5)$$

with $\alpha(V)$ and $\sigma(V)$ sufficiently smooth functions and $W(t)$ a Wiener process. It is assumed at this point that $\alpha(V)$ and $\sigma(V)$ satisfy appropriate conditions which guarantee positivity of the project value for all times. For the Cox-Ingersoll-Ross process such conditions have been given in the previous section, i.e., $2\kappa\theta > \sigma^2$. An agent can then choose a time τ to invest into this project for a sunk investment cost I and by this means to receive a payoff of current value

$$e^{-\rho\tau} (V(\tau) - I), \quad (5.6)$$

where ρ is a discount rate. In doing this the agent's objective is to maximize expected payoff. This leads to an optimal stopping problem

$$F(V) = \max_{\tau} E \left\{ e^{-\rho\tau} (V(\tau) - I) \mid V(0) = V \right\}, \quad (5.7)$$

in which τ can be chosen among all stopping times. At the moment it is assumed that the discount rate is given exogenously. Note that the current model is autonomous and the optimal stopping time has been converted into finding out the value to invest. The current value of the option to invest, i.e. the value of waiting to invest at some later time, when more information on the investment project is revealed, is given by $F(V)$. The option is exercised and investment is undertaken when the investment threshold is reached. This leads to the following condition, which is usually referred to as value-matching condition

$$F(V^*) = V^* - I. \quad (5.8)$$

This condition says that the option is exercised, if and only if its intrinsic value $F(V^*)$ is equal to its current payoff. In order to guarantee certain differentiability properties of the value function, a second so called smooth-pasting condition is made

$$F'(V^*) = 1. \quad (5.9)$$

Finally in the continuation region, i.e., before the option is exercised, the total expected return on the investment opportunity must be equal to its expected rate

of capital appreciation. This gives the so called Bellman equation

$$\rho F dt = E \{dF\}. \quad (5.10)$$

It follows from the Itô formula and equation (5.5) that

$$dF(t) = \alpha(V(t)) \frac{d}{dV(t)} F(V(t)) dt + \frac{\sigma^2(V(t))}{2} \frac{d^2}{d(V(t))^2} F(V(t)) dt + \sigma(V(t)) \frac{d}{dV(t)} F(V(t)) dW(t), \quad (5.11)$$

and therefore (5.10) implies that

$$\frac{\sigma^2(V)}{2} F''(V) + \alpha(V) F'(V) - \rho F(V) = 0. \quad (5.12)$$

Equation (5.12) is a second order ordinary differential equation in which the value function $F(V)$ has to satisfy the two conditions (5.8) and (5.9). In general the solution of a second order ordinary differential equation is uniquely characterized by two independent conditions for F and F' , however, in the current problem the value V^* at which F and F' are determined is also unknown, and this is called a free boundary problem. In order to obtain a unique solution it is therefore necessary to introduce a third condition. The classically used dynamics, geometric Brownian motion and geometric mean reversion techniques, both lead to fixed points at $V = 0$; this condition is usually taken to be

$$F(0) = 0. \quad (5.13)$$

This condition has to be interpreted in a way that, once the value of the investment project has hit zero, it stays zero forever and the option to invest has therefore lost all of its value. This assumption will be discussed further in the framework of the

Cox-Ingersoll-Ross process and the authors will later argue that it makes sense to replace this condition by a finiteness condition

$$F(0) < \infty. \quad (5.14)$$

Furthermore, this will not affect the analysis of the classical cases. To solve the optimal investment problem, it is therefore necessary to solve (5.12) with respect to (5.8), (5.9) and (5.13). Note that in general, and this includes the classical cases, at least one of the equations (5.8) and (5.9) has to be solved numerically. The main critique of the approach presented above is that the discount rate ρ is given exogenously and, in general, it is hard to specify. In the following, the author and his supervisor present an alternative approach to compute the value of the option to invest, which is based on classical contingent claims analysis. This concept was mentioned in section 1.2, and it assumes that there exists a financial asset which is traded on a liquid market, and whose price dynamic is driven by the same noise generating process as the project value, in this case one dimensional Brownian motion $W(t)$. Such an asset is generally referred to as a total spanning asset. It is assumed that the traded asset has a risk adjusted expected rate of return μ and is therefore paying a risk premium of $\mu - r$. According to equation (5.1), the expected rate of return of the investment project is $\frac{\alpha(V)}{V}$. It is important to stress again that the investment project is not understood as a tradable asset. If, however, theoretically it was, then for arbitrage reasons it would need to pay a dividend rate of

$$\delta(V) = \mu - \frac{\alpha(V)}{V}. \quad (5.15)$$

Here it is assumed that the tradable asset has the same volatility structure. The option to invest can now be identified with a perpetual option on this dividend pay-

ing hypothetical stock. The classical approach of constructing a riskless portfolio, which contains the option $F(V)$ and a certain number of shares of stock V , then leads to the following differential equation

$$\frac{1}{2}\sigma^2(V)F''(V) + (r - \delta(V))VF'(V) - rF(V) = 0 \quad (5.16)$$

The two conditions (5.8) and (5.9) remain the same. The advantage of (5.16) is that it does not include the discount rate ρ , which is generally difficult to specify. On the other hand, the contingent claims approach depends on the existence of a total spanning asset. This assumption is in some cases unrealistic. See Ewald and Yang [?] for a utility based approach, which operates in the presence of a partial spanning asset and includes risk aversion toward idiosyncratic risk. These ideas are not, however, followed up in the current chapter.

5.3. The classical cases : Geometric Brownian motion and geometric mean reversion

This section will present the analysis introduced above for the two classical cases where the project value is modelled as a geometric Brownian motion and as a geometric mean reversion process. In the case of geometric Brownian motion,

$$dV(t) = \alpha \cdot V(t)dt + \sigma \cdot V(t)dW(t), \quad (5.17)$$

which in the notation of the previous section gives

$$\alpha(V(t)) = \alpha \cdot V(t), \sigma(V(t)) = \sigma \cdot V(t). \quad (5.18)$$

In this case (5.12) translates to

$$\frac{1}{2}\sigma^2 V^2 F''(V) + \alpha V F'(V) - \rho F = 0. \quad (5.19)$$

Since in the ordinary differential equation above V occurs with the same power as derivatives are taken, it follows that the value function is of the type

$$F(V) = C_1 V^{\beta_1} + C_2 V^{\beta_2}. \quad (5.20)$$

Substitution of this into equation (5.19) leads to

$$\left[\frac{C_1 \beta_1 (\beta_1 - 1) \sigma^2}{2} + C_1 \alpha \beta_1 - C_1 \rho \right] V_1^{\beta_1} + \left[\frac{C_2 \beta_2 (\beta_2 - 1) \sigma^2}{2} + C_2 \alpha \beta_2 - C_2 \rho \right] V_2^{\beta_2} = 0,$$

and then we have

$$\beta_1 = \frac{-\left(\alpha - \frac{\sigma^2}{2}\right) + \sqrt{\left(\alpha - \frac{\sigma^2}{2}\right)^2 + 2\sigma^2 \rho}}{\sigma^2}, \quad (5.21)$$

$$\beta_2 = \frac{-\left(\alpha - \frac{\sigma^2}{2}\right) - \sqrt{\left(\alpha - \frac{\sigma^2}{2}\right)^2 + 2\sigma^2 \rho}}{\sigma^2}, \quad (5.22)$$

which are the two solutions of the so called characteristic equations. Obviously it is assumed that $\beta_2 < 0$ and this guarantees a finite option value at $V = 0$, or, in this case equivalently (5.13) it is necessary to choose $C_2 = 0$. Denoting $C = C_1$ and $\beta = \beta_1$ then gives

$$F(V) = C V^{\beta}, \quad (5.23)$$

where C and V^* are then determined from (5.5) and (5.6). Using the contingent claims approach, it can be shown that the implied proportional dividend rate (5.15) is in fact constant and equations (5.12) and (5.16) are formally identical, with ρ

replaced by r and α replaced by $r - \mu - \alpha$. The solution for $F(V)$ can then simply be obtained from (5.23) and (5.21) by substituting these values. In the case of geometric mean reversion we have

$$dV(t) = \kappa(\theta - V(t))V(t)dt + \sigma V(t)dW(t), \quad (5.24)$$

which in the notation of the previous section corresponds to the choices

$$\alpha(V(t)) = \kappa(\theta - V(t)) \cdot V(t), \quad \sigma(V(t)) = \sigma \cdot V(t), \quad (5.25)$$

and therefore (5.12) translates to

$$\frac{1}{2}\sigma^2 V^2 F''(V) + \kappa(\theta - V)V F'(V) - \rho F(V) = 0. \quad (5.26)$$

From the particular form of this differential equation, it can be assumed that the solution is of the type $F(V) = V^\beta h(V)$. On substitution in (5.26) it can be inferred that

$$\beta_1 = \frac{1}{2} - \frac{\kappa\theta}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\kappa\theta}{\sigma^2}\right)^2 + \frac{2\rho}{\sigma^2}} \quad (5.27)$$

$$\beta_2 = \frac{1}{2} - \frac{\kappa\theta}{\sigma^2} - \sqrt{\left(\frac{1}{2} - \frac{\kappa\theta}{\sigma^2}\right)^2 + \frac{2\rho}{\sigma^2}} \quad (5.28)$$

and

$$F(V) = C_1 V^{\beta_1} h_1(V) + C_2 V^{\beta_2} h_2(V), \quad (5.29)$$

with $h_i(V)$ satisfying

$$\frac{1}{2}\sigma^2 V h_i''(V) + (\sigma^2 \beta_i + \kappa(\theta - V))h_i'(V) - \kappa \beta_i h_i(V) = 0. \quad (5.30)$$

The function $h_i(V)$ can be identified as a Kummer M functions, compared with the discussion in the next section. The Kummer M function however, takes the value 1 at $V = 0$ and therefore, as the exponent β_2 is clearly negative, it is necessary to impose the condition $C_2 = 0$ in order to obtain a finite value for the option to invest at $V = 0$, i.e. condition (5.14) holds. Solving the equation for h_1 then gives

$$F(V) = CV^\beta M\left(\beta, 2\beta + \frac{2\kappa\theta}{\sigma^2}, \frac{2\kappa}{\sigma^2}V\right), \quad (5.31)$$

with $C = C_1$, $\beta = \beta_1$ and $M(a, b, x)$ denoting the Kummer M function which is also known under the name confluent hypergeometric function: see Abramowitz and Stegun (1972) [?]. The values of C and V^* have to be determined from (5.31) in addition to (5.8) and (5.9). Using the contingent claims approach, it is found that in the case of geometric mean reversion the implied proportional dividend rate is given by $\delta(V) = \mu - \kappa(\theta - V)$. In this case equation (5.16) becomes

$$\frac{1}{2}\sigma^2 V^2 F''(V) + \kappa\left(\left(\theta + \frac{r - \mu}{\kappa}\right) - V\right) V F'(V) - r F(V) = 0, \quad (5.32)$$

which is equivalent to (5.26) via the substitution of r for ρ and $\theta + \frac{r - \mu}{\kappa}$ for θ . Using these values in (5.27) and (5.31) leads to

$$F(V) = CV^\beta M\left(\beta, 2\beta + \frac{2(r - \mu + \kappa\theta)}{\sigma^2}, \frac{2\kappa}{\sigma^2}V\right), \quad (5.33)$$

with

$$\beta = \frac{1}{2} + \frac{(\mu - r - \kappa\theta)}{\sigma^2} + \sqrt{\left(\frac{1}{2} + \frac{\mu - r - \kappa\theta}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}. \quad (5.34)$$

5.4. Solution for Cox-Ingersoll-Ross type project value

In this section, it is assumed that the dynamic project value is given by a Cox-Ingersoll-Ross process whose dynamic is given by (5.1). It is believed that this process has never been used before in the theory of real options and irreversible investment, even though it has significant advantages compared to the classical choices of geometric Brownian motion and geometric mean reversion. This was discussed in section 5.1 and will be expanded upon in the next section. In the notation of section 5.2

$$\alpha(V(t)) = \kappa(\theta - V(t)), \sigma(V(t)) = \sigma\sqrt{V(t)}. \quad (5.35)$$

This has two fundamental differences when compared to the geometric mean reversion process (2.24). Firstly, the mean reversion speed is not level dependent and secondly, the variance is proportional to the current level $V(t)$ rather than the standard deviation. Applying the general methodology introduced in section 1.2 produces

$$\frac{\sigma^2 V}{2} F''(V) + \kappa(\theta - V) F'(V) - \rho F(V) = 0. \quad (5.36)$$

The value function of the irreversible investment problem $F(V)$ must therefore satisfy (5.36) with respect to the value-matching and smooth-pasting conditions, i.e., equations (5.8) and (5.9). The following section will show explicitly how to solve this differential equation. Dividing equation (5.36) by $\frac{1}{k}$ and using the following transformation

$$a := \frac{\rho}{\kappa}, b := \frac{2\kappa\theta}{\sigma^2}, z := \frac{2\kappa}{\sigma^2} v, \quad (5.37)$$

and furthermore writing $F(v) = w\left(\frac{2\kappa}{\sigma^2}v\right)$ leads to the following differential equation for $w(z)$:

$$zw''(z) + (b - z)w'(z) - aw(z) = 0 \quad (5.38)$$

This is the well known Kummer equation. It has two fundamental solutions from which all solutions can be obtained by linear combinations. These are the Kummer M function and the Kummer U function. These are classically denoted with $M(a, b, z)$ for Kummer M and $U(a, b, z)$ for Kummer U . The Kummer U function can be expressed in terms of the Kummer M and its use can be avoided, see [?], equation 13.1.3. From this, the general solution to equation (5.36) is given by

$$F(V) = C_1 M\left(\frac{\rho}{\kappa}, \frac{2\kappa\theta}{\sigma^2}, \frac{2\kappa}{\sigma^2}V\right) + C_2 U\left(\frac{\rho}{\kappa}, \frac{2\kappa\theta}{\sigma^2}, \frac{2\kappa}{\sigma^2}V\right). \quad (5.39)$$

The Kummer M and U functions show the following extremal behaviour for the limit $V \rightarrow 0$

$$\lim_{z \rightarrow 0} M(a, b, z) = 1 \quad (5.40)$$

$$\lim_{z \rightarrow 0} U(a, b, z) = \infty \quad (5.41)$$

From this it becomes clear that the condition $F(0) = 0$ cannot be satisfied in any case but the trivial one $C_1 = C_2 = 0$. This choice, however, would not satisfy the value-matching and smooth-pasting condition and, furthermore, it would be economically unreasonable. It is therefore necessary to relax this condition. In terms of the discussion at the beginning, this is perfectly natural and economically sound. The option value at $V = 0$ must be positive as the dynamic of the Cox-Ingersoll-Ross process at $V = 0$ drives the project to a strictly positive value in an instantaneous amount of time. Note that formally substituting the value $V(t) = 0$

into the dynamic (5.1) gives a strictly positive drift of $\kappa\theta dt$. This behaviour of the Cox-Ingersoll-Ross process is fundamentally different than in the classical cases. In order to avoid $F(0) = \infty$ which is economically unreasonable, it is necessary to choose $C_2 = 0$. It can be seen that, by working with the Cox-Ingersoll-Ross process as a model for the project value, the classical condition (5.13) needs to be replaced with condition (5.14). It was demonstrated in section 5.3 that this replacement does not have any effect on the solution of the classical cases. In the classical cases, (5.14) in fact implies (5.13) and therefore it is proposed to make this adjustment in general. The property of the current model that the option value at $V = 0$ is not zero however clearly distinguishes the Cox-Ingersoll-Ross based irreversible investment problem from the classical ones. In the author and his supervisor's opinion, this makes the assumption of a Cox-Ingersoll-Ross process much more realistic, when the project is able to resurface after it has hit bankruptcy. It is now necessary to choose the parameters C_1 and the investment threshold V^* such that the smooth-pasting and value-matching conditions are satisfied. Starting with the smooth-pasting condition (5.9), this condition leads to the equation

$$C_1 = \frac{\kappa\theta}{\rho M\left(\frac{\rho}{\kappa} + 1, \frac{2\kappa\theta}{\sigma^2} + 1, \frac{2\kappa}{\sigma^2}V^*\right)}. \quad (5.42)$$

Solving for C_1 and substituting this value in (5.39) gives

$$F(V) = \frac{\kappa\theta M\left(\frac{\rho}{\kappa}, \frac{2\kappa\theta}{\sigma^2}, \frac{2\kappa}{\sigma^2}V\right)}{\rho M\left(\frac{\rho}{\kappa} + 1, \frac{2\kappa\theta}{\sigma^2} + 1, \frac{2\kappa}{\sigma^2}V^*\right)} \quad (5.43)$$

with V^* the unique solution of (5.8). In common with the case of geometric mean reversion where the Kummer M function occurs as well, it is not possible to solve the value-matching equation analytically; instead, it is necessary to use numerical methods. This, however, is no more challenging than in the geometric

mean reversion case. Moving on to the case where the option to invest is evaluated with the contingent claims approach, in this case the resulting differential equation for the option value is

$$\frac{\sigma^2 V}{2} F''(V) + [\kappa(\theta - V) + (r - \mu)V] F'(V) - rF(V) = 0. \quad (5.44)$$

Denoting

$$\tilde{\kappa} := \kappa + \mu - r \quad (5.45)$$

$$\tilde{\theta} := \frac{\kappa\theta}{\kappa + \mu - r}, \quad (5.46)$$

it can be seen that (5.44) is identical to (5.36) where κ and θ are replaced by $\tilde{\kappa}$ and $\tilde{\theta}$ and ρ is replaced by r . The value function $F(V)$ for the option to invest is therefore given by

$$F(V) = CM \left(\frac{r}{\kappa + \mu - r}, \frac{2\kappa\theta}{\sigma^2}, \frac{2(\kappa + \mu - r)}{\sigma^2} V \right) \quad (5.47)$$

with C given by

$$C = \frac{\kappa\theta}{rM \left(\frac{r}{\kappa + \mu - r} + 1, \frac{2\kappa\theta}{\sigma^2} + 1, \frac{2(\kappa + \mu - r)}{\sigma^2} V^* \right)} \quad (5.48)$$

and V^* determined by (5.9). As the Kummer M function gives the value 1 for $V = 0$, C is identified as the value of the option at $V = 0$.

5.5. Qualitative and Quantitative differences to the classical choices

This section presents several numerical examples and it can be seen that the Cox-Ingersoll-Ross process is a more natural option within a real option framework

than the geometric mean reversion process. In order to achieve this, numerical results for the geometric mean reversion dynamic (5.24) will be reproduced, as obtained by Dixit and Pindyck (1994) [?], Figures 5.11-5.17, pages 164-170. The same parameters will then be used with the Cox-Ingersoll-Ross process dynamics defined by (5.1), and in this way it is possible to make precise comparisons and highlight the differences between the two approaches. Note that Dixit and Pindyck used a slightly different notation in which κ and θ map to η and \bar{V} , respectively, and the common parameters which remain unchanged in all cases are the interest rate $r = 0.04$, the volatility parameter $\sigma = 0.2$ and the sunk cost $I = 1$. The following observations relate to the figures in Graphical Illustration. Odd numbered figures correspond to the Cox-Ingersoll-Ross process, while even numbered figures refer to the geometric mean reversion process.

Figures 5.1 to 5.6 show the dependence of the value of the option to invest $F(V)$ with respect to the project value V . It can be seen that, in all cases, a higher value of the mean reversion parameter θ leads to a higher value of the option and, furthermore, that the value of the option is increasing with regard to the project value V . This makes sense from an economic point of view. For a rather small mean reversion speed $\kappa = 0.05$ the value functions in Figure 5.1 and 5.2 appear, at least from a qualitative point of view, to be very similar. A difference that can be observed is that the graphs in Figure 5.1 all start from a strictly positive value while in Figure 5.2, they all start at 0. This is expected from the analysis in section 5.1, 5.3 and 5.4. The same behaviour can be observed in Figure 5.3 and 5.4 where the mean reversion speed is increased to $\kappa = 0.1$. For the higher mean reversion level considered here, $\theta = 1.5$, however, it can now be seen that, in the case of geometric mean reversion, i.e., Figure 5.4, the value function has a slightly awkward bend and is qualitatively very different than the corresponding

value function for the Cox-Ingersoll-Ross process. Increasing the mean reversion speed further to $\kappa = 0.5$ reinforces this effect. Note in Figure 5.6, the graph for the value function still starts at $F(0) = 0$ and with an almost infinite slope to attain a similar level to the value function for the Cox-Ingersoll-Ross analogue. Within the Cox-Ingersoll-Ross framework, i.e. Figure 5.5, the value function starts at a finite positive level and has an upward bend. Further raising the speed of mean reversion increases this characteristic. The result is that, in the geometric mean reversion framework, the value of the option to invest is extremely sensitive with respect to small changes, when the value of the project is low. In both the author and his supervisor's opinion, it is much too sensitive, even with reasonable parameter choices. In contrast, the value function in the Cox-Ingersoll-Ross framework shows a very regular behaviour for small project values and should therefore be considered to be the preferred choice.

Figures 5.7 to 5.10 display the dependence of the investment threshold V^* with respect to the mean reversion speed κ in each of the two cases, for high and low return rate μ of the hedging asset. Figure 5.7 and 5.9 only display the range starting from $\kappa = 0.1$ since the Cox-Ingersoll-Ross dynamic is only well defined under the condition $2\kappa\theta > \sigma^2$ and lower values cause problems with the numerical solution routine. It can be seen that investment thresholds in the geometric mean reversion model are generally higher than in the Cox-Ingersoll-Ross framework. A closer inspection of Figure 5.8 and 5.10 shows that in the case of a high mean reversion level $\theta = 1.5$ the critical value V^* does not demonstrate monotonic behaviour. In the parameter range studied, the effect is, however, rather insignificant. More significant is the common property between the Cox-Ingersoll-Ross framework and the geometric mean reversion framework, that depending on whether the current value is higher or lower than the mean reversion level, θ , the investment threshold

is initially decreasing or increasing with respect to the mean reversion speed κ .

Figures 5.11 to 5.14 display the dependence of the investment threshold on the return rate of the hedging asset. Here the qualitative behavior in the two case Cox-Ingersoll-Ross and geometric mean reversion is very similar, but again the investment threshold under the Cox-Ingersoll-Ross dynamic is significantly lower than under the geometric mean reversion dynamic. In all cases the investment threshold V^* is decreasing with the return rate μ . Economically, this can be explained by the fact that increases in μ lead to increases in the implied proportional dividend yield (5.15), and increasing the dividend yield provides an incentive to exercise the option earlier, therefore lowering V^* .

5.6. Conclusions

An explicit solution has been derived for the value function for the option of irreversible investment into a project whose value is modelled as a Cox-Ingersoll-Ross process. The results have been compared with the classical ones obtained for the cases of geometric Brownian motion and geometric mean reversion. This includes a numerical analysis and graphical illustrations. The various advantages of the Cox- Ingersoll-Ross process with respect to the geometric mean reversion process have been examined and discussed.

Graphical Illustration

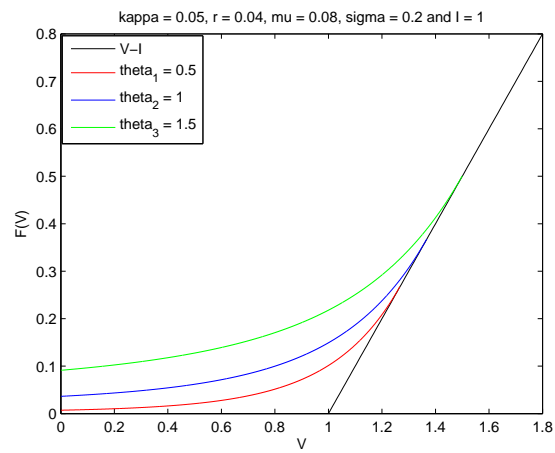


FIGURE 5.1: CIR

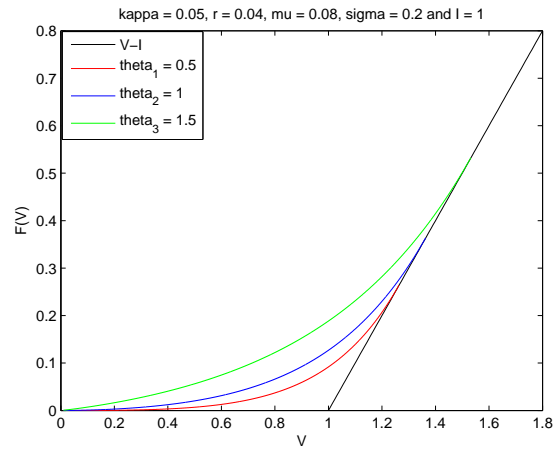


FIGURE 5.2: GMR

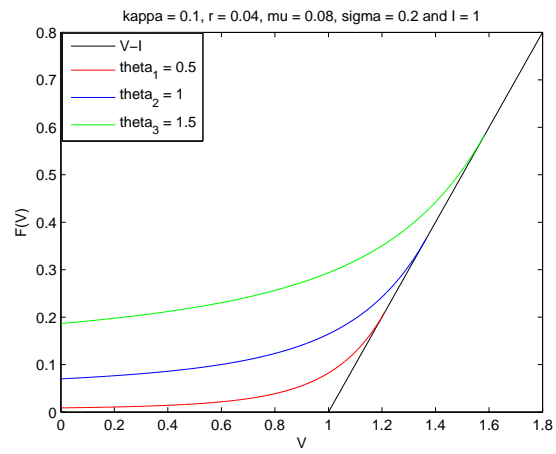


FIGURE 5.3: CIR

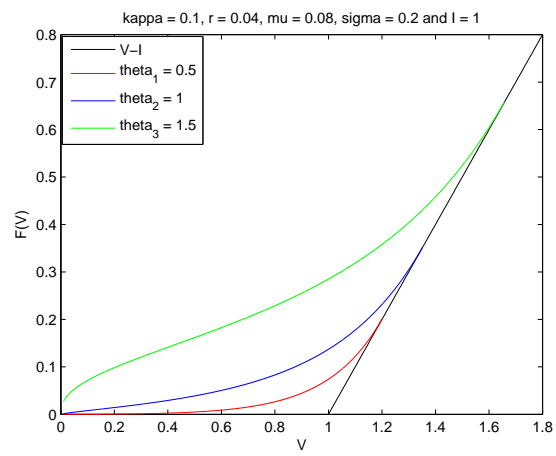


FIGURE 5.4: GMR

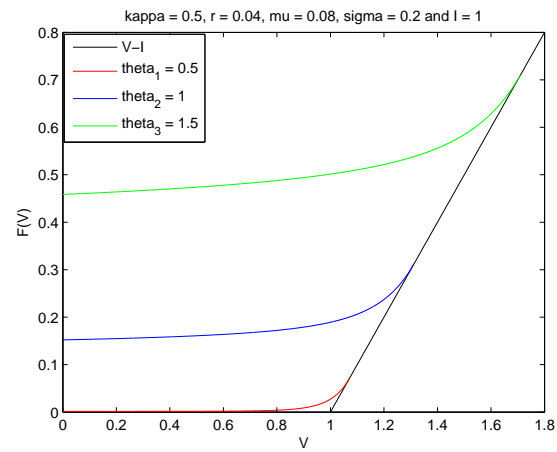


FIGURE 5.5: CIR

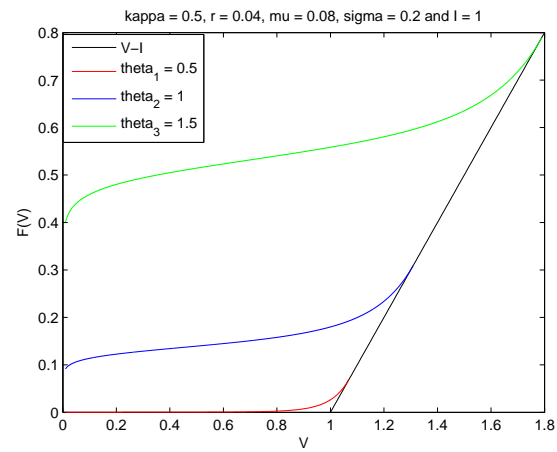


FIGURE 5.6: GMR

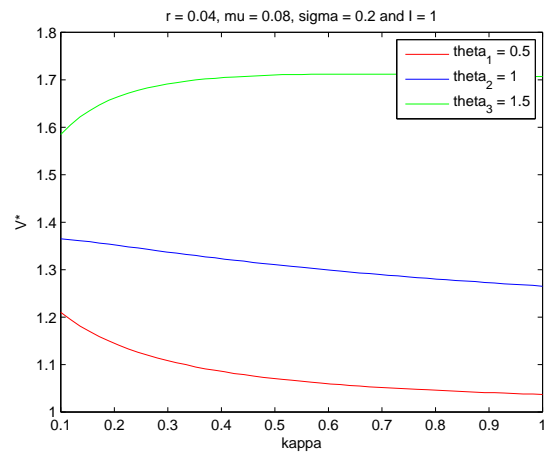


FIGURE 5.7: CIR

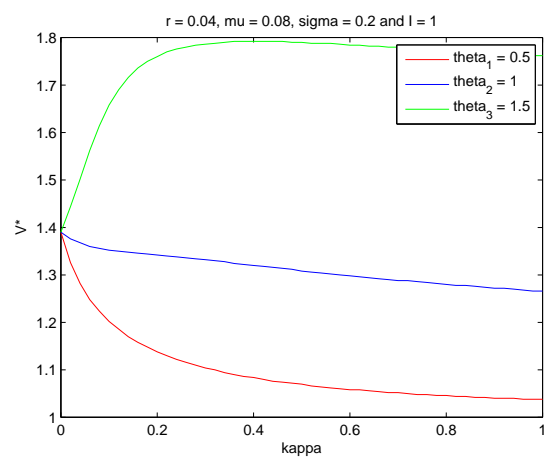


FIGURE 5.8: GMR

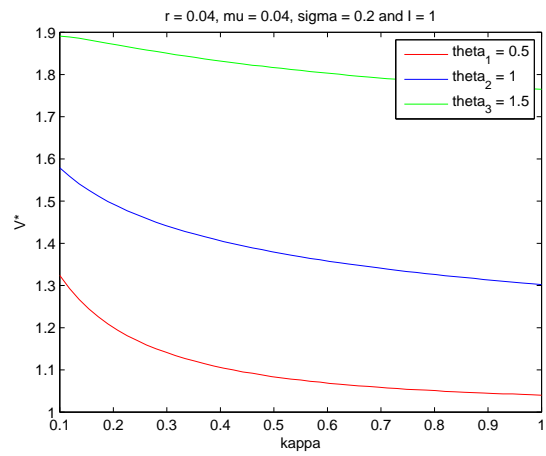


FIGURE 5.9: CIR

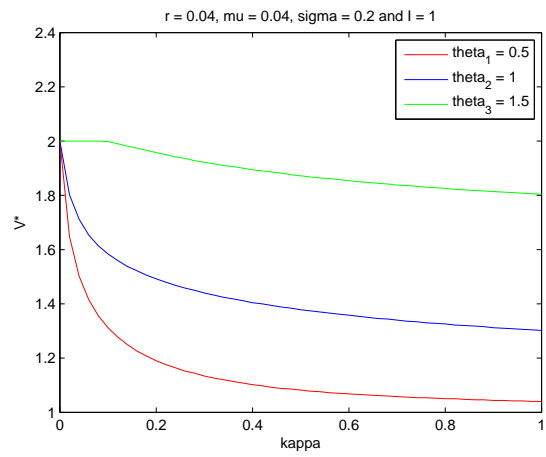


FIGURE 5.10: GMR

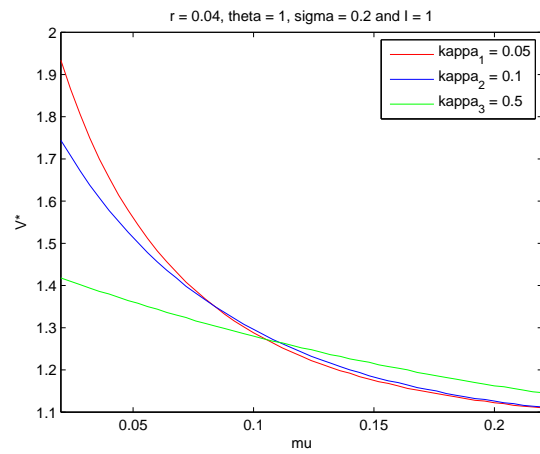


FIGURE 5.11: CIR

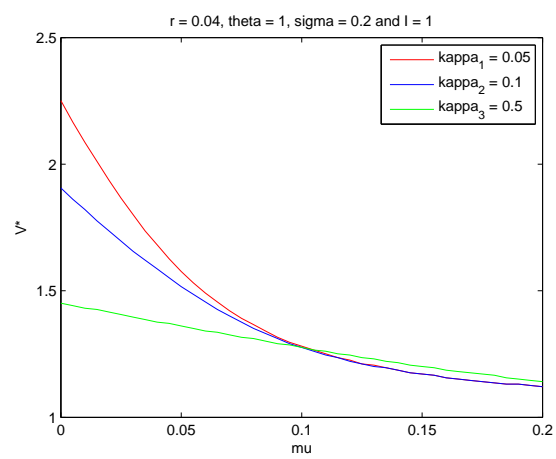


FIGURE 5.12: GMR

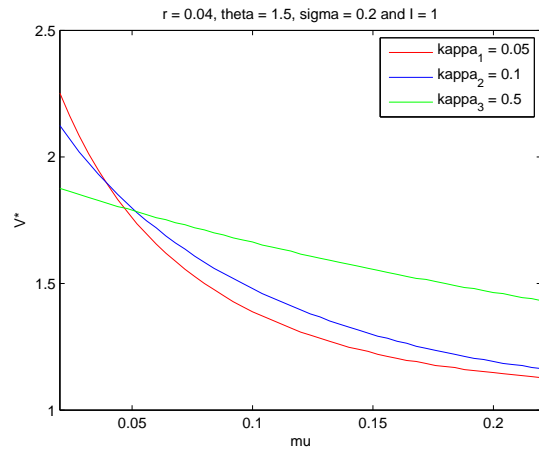


FIGURE 5.13: CIR

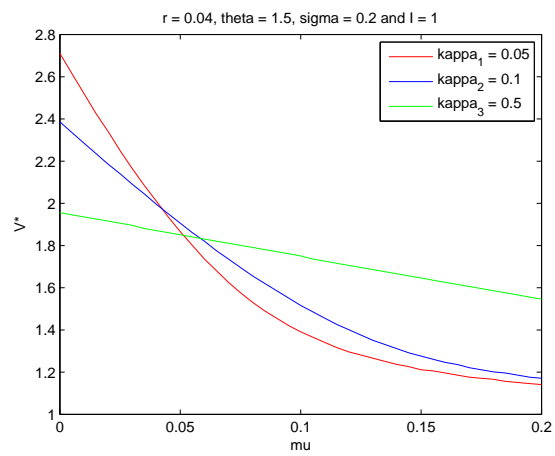


FIGURE 5.14: GMR

Chapter 6

Conclusions and Future Work

This thesis presents several applications of the differential game theory and real option approaches in environmental economics. Note that all models presented in the thesis are theoretical models, not empirical models. Differential game theory has been used to study games defined in continuous time and a differential game model contains objective functionals for each player and states equations given by some differential equations. On the other hand, due to the role played by uncertainty, it is possible to consider the state equations as stochastic differential equations, which allows study of how uncertainty affects the models. The technique used to solve the stochastic differential games is the Hamilton-Jacobi-Bellman equations, which lead to a system of differential equations. Differential game theory is widely applied in several fields such as management, finance and economics.

The applications of stochastic differential games presented in this thesis are public goods and fisheries. In Chapter 2, the author has considered a public good whose property is that suffering is caused by it if the value is sufficiently high. One example of this is over-development of environmental resources. Two types of uncertainty have been analyzed and it has been demonstrated that higher risk reinforces the free

ride effect, which is opposite to the prediction that increased uncertainty reduces free rider effect. Since each agent's revenue is offset by a higher uncertainty, it leads to a more apparent free ride effect.

On the other hand, over-conservation of environmental resources may lead economic development to be hampered and it leads to one extension of Chapter 2. Stratford (2008) studied the case of the island of Tasmania in Australia and notes "Conflicts over conservation and development emerged again over the period from 1989 to 1994 when the Tasmanian State Government, led by Premier Michael Field, confronted a fiscal crisis and was subject to intense local pressures to embrace the new international rhetoric of sustainable development as it had been conceived in the Brundtland Report (World Commission on Environment and Development, 1987) and gained momentum via Australian Government strategies for ecological sustainable development", see [?]. Therefore, when considering development, governments and individuals should also be concerned with conservation, i.e., it is important to avoid both over-development and over-conservation.

In Chapters 3 and 4, the fishery problem has been studied. This thesis analyses how market prices, individual costs and the interactions between species affect the harvest rates for fishery agents. Two different cases of ecological system have been examined, i.e. a competitive, and a predator-prey system. Different competitive arrangements between the fishery agents have also been analysed, i.e. the case where fishery agents are competing, where they are allowed to catch a single species, and where they cooperate. In the case of the competitive ecological system, the results demonstrate that, both economically and ecologically, cooperation may be a better option, while in a predator-prey system, cooperation may lead species to become extinct earlier. It can be seen that under some conditions the prey may become extinct easily, which in turn causes the predator to lack food and

also become extinct. In the model presented, it was assumed that the price for each species was not dependent on the other species. This may not be realistic in some cases, but this simplification makes it possible to solve the model semi-analytically. The numerical example generated shows that, under a predator-prey system, both species become extinct. Therefore, to provide data to support the possibility of policy management, the stochastic version of maximum sustainable yield is developed in Chapter 4. Several different cases have been analyzed and it has been identified that the impact of a sufficiently high uncertainty always leads to a lower optimal harvest rate.

Fisheries have suffered from climate change and the model presented identifies how fishermen choose harvest rates under conditions of higher uncertainty. On the other hand, some regulatory policies have already been introduced by governments such as imposing a tax on fish caught and individual quota management. These policies lead to some extensions of Chapter 3 and 4. The former has been examined by Pradhan and Chaudhuri (1999) [?], where they studied a deterministic model in which the regulatory agent imposes a tax to control the harvest rates. Arnason, on the other hand, introduced the idea of an individual quota system and adopted it in 1975, the first in the world. Quota systems have been applied in many countries such as the United States, Australia and the Republic of Chile. The mechanism used involves the allocation of quotas to each fishery and the sum of all quotas is less than the total allowable catch (TAC). Agents are allowed to trade the quotas in a market. One advantage of quota systems is that agents have more incentive to invest in their equipment so that they catch fish more efficiently and reduce their costs. More details can be found in the report by Hatcher, Pascoe, Banks and Arnason (2002) [?]. Arnason also presented a paper in 1998 in which he studied a continuous model with a single agent and adopted the individual transferable share

quota system, see [?]. In this system the agents trade shares in the market instead of the number of quotas. Since the sum of quotas is limited and there should be several agents in the market, it would be worthwhile investigating a game model of an individual transferable quota system.

The real option approach is used to derive an optimal stopping time for an irreversible investment. Two examples of an irreversible investment are investing in undeveloped oil reserves and adopting new policies. The former example involves a huge amount of capital and firms have to consider the optimal timing to invest in undeveloped oil reserves, while in the latter example, authorities are concerned about the fact that the policies affect both development and conservation. These policies, therefore, lead authorities to consider the optimal timing to adopt these policies. It has been proposed that the project value follows a stochastic differential equation and then the Hamilton-Jacobi-Bellman equation leads to a differential equation. This differential equation is associated with two conditions, namely value-matching and smooth-pasting conditions, which leads to a free boundary value problem.

In Chapter 5, the real option approach was applied to study the case of a Cox-Ingersoll-Ross process which follows the form,

$$dx(t) = \kappa (\theta - x(t)) dt + \sigma \sqrt{x(t)} dW(t),$$

under the scenario in [?], Chapter 5. A Cox-Ingersoll-Ross process represents a project which can recover, i.e. if the value reaches 0, it increases immediately. This occurs, for example, with the price of oil. Even though the stock of oil is reduced permanently, the stock so far is still sufficient to meet demand and represents one of the world's main resources. The example given in [?], Chapter 12 C, is based on a geometric mean reverting process. Since such processes remain at zero if zero is

ever achieved, it is worth considering that the price of some non renewable resources are defined by a Cox-Ingersoll-Ross process. The reason for this is that the main resources are still needed and, if the price drops too low, it would be expected to increase. It has been demonstrated that, in the case of a Cox-Ingersoll-Ross process, the value around 0 is regular while in the case of a geometric mean reverting process, the value increases dramatically. Such phenomenon may not be realistic.

On the other hand, real option theory can be applied to other problems in environmental economics. One example is anti-pollution. Two examples of such a problem have been reported. In 2006, over a thousand villagers from the village of Mehdiganj in the north Indian state of Uttar Pradesh protested at the headquarters of the Coca-Cola Company in Gurgaon because of polluted groundwater and soil. The second illustration refers to an event that took place in China in 2005. According to a report by AsiaNews, see [?], 60,000 Chinese people in Huaxi Village, Zhejiang Province protested in April 2005 against high levels of pollution emitted by 13 chemical plants located in the area. Real option theory can be applied to find the optimal timing when a firm should adopt the pollution abatement facilities. Such an application also provides governments several policies how to help the firms to invest in the pollution abatement facilities earlier.

The models presented in this thesis are either based on the real option theory or a differential game due to the requirements of mathematical tractability. However, some applications such as exploitation of environmental resources could be relevant to both techniques. For example, the Exclusive Economic Zone (EEZ) of the Republic of Kiribati is around 3.5 million km^2 , which is a significant tuna fishing zone for industrial fleets from a number of distant-water fishing nations (DWFNs) including Japan, Taiwan, Korea, the United States, and Spain. The current situation is that foreign fishery companies buy licenses to catch fish in the fishery zone and this is

one of the major sources of revenue for the country. Instead of selling licenses, the government could consider the idea of imposing a tax on fish caught by foreign companies. To catch fish in Kiribati, the foreign company is required to invest in equipment such as vessels and warehouses in Kiribati. In this case, a foreign company has to concentrate on two aspects, i.e., the optimal timing when it should invest in the country and the harvest rate. The former represents a good case for applying the real option approach, while the latter could be a differential game among the government and all fishery agents. Such applications allow combination of both techniques in order to study more realistic models.

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